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The Saff-Varga Width Conjecture and entire functions with simple exponential growth

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Abstract We show that the partial sums of the power series for a certain class of entire functions possess scaling limits in various directions in the complex plane. In doing so we obtain information about the zeros of the partial sums. We will only assume that these entire functions have a certain asymptotic behavior at infinity.

With this information we will partially verify for this class of functions a conjecture on the location of the zeros of their partial sums known as the Saff-Varga Width Conjecture.

Numerical results and figures are included to illustrate the results obtained for several well-known functions including the Airy functions and the parabolic cylinder functions.

Keywords Taylor polynomials · asymptotic analysis · scaling limits · zeros

Mathematics Subject Classification (2000) 30B10 · 30E15 · 30D15

1 Introduction

In their 1976 paper [21] Saff and Varga made the following conjecture (see also [22,4]).

Saff-Varga Width Conjecture. *Consider the "parabolic region"*

$$S_0(\tau) = \left\{ z = x + iy : |y| \leq K x^{1-\tau/2},\, x \geq x_0 \right\},$$

where K and x_0 are fixed positive constants, and consider also the regions $S_\theta(\tau)$ obtained by rotations of $S_0(\tau)$:

$$S_\theta(\tau) = e^{i\theta} S_0(\tau).$$

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Given any entire function f of positive finite order $\lambda > \tau$, denote its n^{th} partial sum by $p_n(z)$. There exists an infinite sequence M of positive integers such that there is no $S_\theta(\tau)$ which is devoid of all zeros of all partial sums $p_m(z)$, $m \in M$.

Essentially what this conjecture is saying is that if an entire function of positive, finite order has a zero-free region, then that region may not be 'too wide'—this width depending on the order of the function.

According to Edrei, Saff, and Varga, their 1983 monograph [4] arose from an attempt to settle this conjecture. In it the authors studied the partial sums of the Mittag-Leffler function $E_{1/\lambda}$ for $\lambda > 1$. The function $E_{1/\lambda}$ can be considered a generalization of the exponential function e^z and is defined for $z \in \mathbb{C}$ and $\lambda > 0$ by

$$E_{1/\lambda}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k/\lambda + 1)}.$$

When $\lambda = 1$ the usual exponential function is recovered, and in general $E_{1/\lambda}$ is an entire function of order λ.

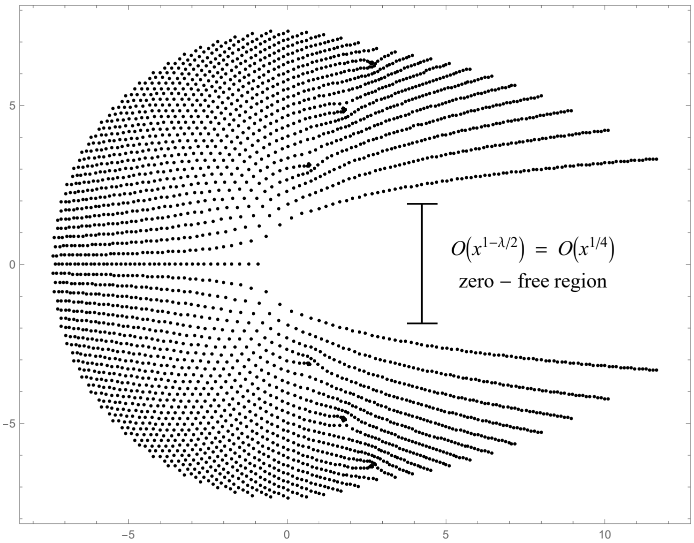

Fig. 1 Zeros of the partial sums $p_n[E](z)$ for $n = 1, \ldots, 80$ with $\lambda = 3/2$. A zero-free region can be seen opening to the right whose width is in agreement with the Saff-Varga Width Conjecture.

Let

$$p_n[E](z) = \sum_{k=0}^{n} \frac{z^k}{\Gamma(k/\lambda + 1)}$$

denote the n^{th} partial sum of $E_{1/\lambda}$. Edrei, Saff, and Varga found that the zeros of $p_n[E]$ which do not converge to zeros of $E_{1/\lambda}$ grow at a rate comparable to

$$r_n := e^{1/(2n)} \left(\frac{n}{\lambda}\right)^{1/\lambda},$$

and, more precisely, that the corresponding zeros of the scaled partial sums $p_n[E](r_n z)$ converge to the curve

$$S_E = \left\{ z \in \mathbb{C} : \left| z^\lambda \exp\left(1 - z^\lambda\right) \right| = 1,\ |z| \leq 1,\ \text{and } |\arg z| \leq \frac{\pi}{2\lambda} \right\} \\ \cup \left\{ z \in \mathbb{C} : |z| = e^{-1/\lambda} \text{ and } |\arg z| \geq \frac{\pi}{2\lambda} \right\}.$$

In particular the authors proved the following two theorems which give detailed information about how the zeros of the partial sums approach this curve S_E.

Theorem 1 (Edrei, Saff, and Varga)

$$\lim_{n\to\infty} \frac{p_n\left(r_n\left(1 + w\sqrt{2/(\lambda n)}\right)\right)}{\left(1 + w\sqrt{2/(\lambda n)}\right)^n E_{1/\lambda}(r_n)} = \frac{1}{2}\exp\left(w^2\right)\operatorname{erfc}(w)$$

uniformly for w restricted to any compact subset of $\mathbb{C}$.

The function erfc here is known as the complementary error function and is defined by

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-s^2}\, ds,$$

where the contour of integration is the horizontal line starting at $s = z$ and extending to the right to $s = z + \infty$. This function has infinitely many zeros, all of which lie in the sectors $\pi/2 < |\arg z| < 3\pi/4$ and approach asymptotically the rays $\arg z = \pm 3\pi/4$ (see, e.g., [5]).

Theorem 2 (Edrei, Saff, and Varga) *Let $\xi \in S_E$ with $0 < |\arg \xi| < \pi/(2\lambda)$ be fixed and set*

$$\tau = |\xi|^\lambda \sin(\lambda \arg \xi) - \lambda \arg \xi.$$

Define the sequence (τ_n) by the condition

$$\frac{\tau}{\lambda} n \equiv \tau_n \pmod{2\pi}, \qquad -\pi < \tau_n \leq \pi.$$

Let

$$\zeta_0 = -\frac{1}{2}\log(2\pi\lambda) + \frac{1}{2}\left(1 - \xi^\lambda\right) + \log\left(\frac{\xi}{1-\xi}\right),$$

where $\log(2\pi\lambda)$ is real and the determination of the last logarithm is such that

$$-\pi < \operatorname{Im}\log\left(\frac{\xi}{1-\xi}\right) \leq \pi.$$

Put

$$P_n = \left(1 + \frac{\log n + 2i\tau_n - 2\zeta_0}{2(1-\xi^\lambda)n}\right) r_n \xi$$

and consider all zeros of the polynomial in ζ given by

$$\psi_n(\zeta) := p_n[E]\left(P_n - \frac{r_n \xi}{(1-\xi^\lambda)n}\zeta\right),$$

where r_n and $p_n[E]$ are as in Theorem 1. Given $t > 0$ (t not a multiple of 2π), the polynomial ψ_n has, in the disk $|\zeta| \le t$, exactly

$$2\left\lfloor \frac{t}{2\pi} \right\rfloor + 1 =: 2\ell + 1$$

zeros, all of them simple. Denoting these zeros of ψ_n by $\zeta_{n,j}$, $j = 0, \pm 1, \ldots, \pm \ell$, then

$$\zeta_{n,j} = 2j\pi i + \eta_{n,j}, \qquad j = 0, \pm 1, \ldots, \pm \ell,$$

where for fixed t

$$\lim_{n\to\infty} \eta_{n,j} = 0.$$

A similar result holds for $\xi \in S_E$ with $|\arg \xi| > \pi/(2\lambda)$.

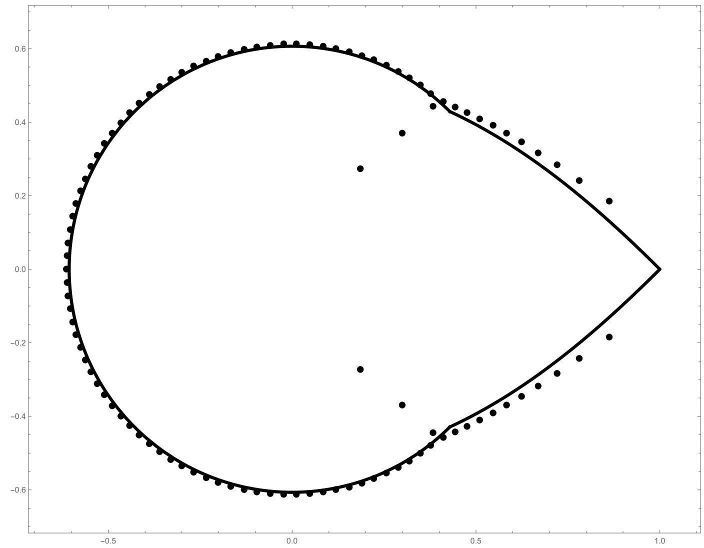

Fig. 2 The zeros of the **scaled** partial sums $p_{105}[E](r_{105}z)$ with $\lambda = 2$, shown with the curve S_E.

In the context of Hurwitz's theorem these two results tell us that the scaled partial sums $p_n[E](r_n z)$ of the Mittag-Leffler function have zeros which approach points $\xi \in S_E$ with $\xi \ne 1$ and $|\arg \xi| \ne \pi/(2\lambda)$ at a rate of $\Theta(\log n/n)$ and which are separated from each other by a distance of $\Theta(n^{-1})$, and zeros which approach the point $\xi = 1$ at a rate of $\Theta(n^{-1/2})$ and which are separated by a distance of $\Theta(n^{-1/2})$. In fact this behavior is typical of all of the entire functions which have been studied to date: informally, most of the zeros of the partial sums cluster densely together and "fill" up most of the plane, and there are only a finite number of exceptional arguments where the zeros are widely spaced and zero-free regions like the one shown in Figure 1 exist.

To capture these observations, Edrei, Saff, and Varga proposed a modified version of the Width Conjecture [4, p. 6].

Modified Saff-Varga Width Conjecture. *Let f be an entire function of positive, finite order λ and let $p_n(z)$ denote its n^{th} partial sum. We can find an infinite sequence M of positive integers and a finite number of exceptional arguments $\theta_1, \theta_2, \ldots, \theta_q$ such that*

(a) *For any argument $\theta \neq \theta_j$, $j = 1, 2, \ldots, q$, it's possible to find a positive sequence $(\rho_m)_{m\in M}$ with $\rho_m \to \infty$ and $\rho_m = O(m^{2/\lambda})$ such that, for every fixed $\epsilon > 0$, the number of zeros of the partial sum $p_m(z)$ in the disk*

$$\left|z - \rho_m e^{i\theta}\right| \leq \rho_m m^{-1+\epsilon}$$

tends to infinity as $m \to \infty$, $m \in M$.

(b) *For any exceptional argument θ_j it's possible to find an integer $k \geq 2$ and a positive sequence $(\rho_m)_{m\in M}$ with $\rho_m \to \infty$ and $\rho_m = O(m^{2/(\lambda k)})$ such that, for every fixed $\epsilon > 0$, the number of zeros of the partial sum $p_m(z)$ in the disk*

$$\left|z - \rho_m e^{i\theta_j}\right| \leq \rho_m m^{-1/k+\epsilon}$$

tends to infinity as $m \to \infty$, $m \in M$.

One can check that a verification of the Modified Width Conjecture would imply the truth of the standard Width Conjecture.

For the particular case of the Mittag-Leffler function, the scaling limit in Theorem 1 verifies part (b) of the Modified Width Conjecture at the exceptional argument $\theta = 0$ with $k = 2$ and Theorem 2 verifies part (a) along any argument $\theta \neq 0, \pm\pi/(2\lambda)$. Together with the fact that the zeros of $E_{1/\lambda}$ lie asymptotically on the rays $\arg z = \pm\pi/(2\lambda)$ these results imply that the original Saff-Varga Width Conjecture holds for this function. The authors also proved versions of Theorems 1 and 2 for $\mathcal{L}$-functions [4, p. 21], likewise verifying the Width Conjecture for those functions.

Prior to Edrei, Saff, and Varga's work, Newman and Rivlin derived the following scaling limit for the partial sums of the exponential function.

Theorem 3 (Newman and Rivlin [14]) *Let*

$$p_n[\exp](z) = \sum_{k=0}^{n} \frac{z^k}{k!}$$

denote the n^{th} partial sum of the exponential function. Then

$$\lim_{n\to\infty} \frac{p_n[\exp](n + w\sqrt{n})}{\exp(n + w\sqrt{n})} = \frac{1}{2}\operatorname{erfc}\left(\frac{w}{\sqrt{2}}\right)$$

uniformly for w restricted to any compact subset of $\mathbb{C}$.

This limit serves to verify part (b) of the Modified Width Conjecture at the exceptional argument $\theta = 0$ with $k = 2$ in the particular case of the usual exponential function just as Theorem 1 did in the case of the Mittag-Leffler function with $\lambda > 1$.

Later Norfolk obtained the following analogue of Theorems 1 and 3 which verifies the Modified Width Conjecture at the exceptional argument $\theta = 0$ with $\lambda = 1$, $\rho_n = n$, and $k = 2$ in the case of the confluent hypergeometric function.

Theorem 4 (Norfolk [15]) *Let*

$$ {}_1F_1(1;b;z) = \Gamma(b) \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+b)} $$

and let $p_n[{}_1F_1](z)$ *denote its* n^{th} *partial sum. If* $b \in \mathbb{R}$ *with* $b \neq 1, 0, -1, -2, \ldots$ *then*

$$ \lim_{n\to\infty} \frac{p_n[{}_1F_1](n + w\sqrt{n})}{e^{w\sqrt{n}}\, {}_1F_1(1;b;n)} = \frac{1}{2} \operatorname{erfc}\left(\frac{w}{\sqrt{2}}\right) $$

uniformly for w *restricted to any compact subset of* $\mathbb{C}$.

Other results in this same vein have been proved for binomial expansions [7], for linear combinations of sections and tails of Mittag-Leffler functions [28] and of classical Lindelöf functions [18], and most recently for Riemann's Xi function [8].

The content of Edrei, Saff, and Varga's monograph [4] inspired much of the work that appears in this paper, and the Mittag-Leffler function $E_{1/\lambda}$ is essentially the archetype of the functions we will consider. This paper is an attempt to make progress toward a resolution of the Width Conjecture by proving general versions of Theorems 1, 2, 3, 4, and related, thereby verifying the Modified Saff-Varga Width Conjecture for the wide class of functions we consider in their sectors of maximal growth. We discuss this further in Subsection 1.2.

1.1 A Riemann-Hilbert approach

In 2008 Kriecherbauer, Kuijlaars, McLaughlin, and Miller published a paper [10] in which they undertook an analysis of the partial sums of the exponential function using a version of the Riemann-Hilbert method of asymptotic analysis. Among other things the authors obtained a complete asymptotic expansion of $p_n[\exp](nz)$ in the regime $n \to \infty$, $z \to 1$.

They began by defining the function

$$ F_n(z) = \frac{1}{2\pi i} \int_\gamma \frac{(se^{1-s})^{-n}}{s-z}\, ds, $$

where γ is a simple closed loop around the origin passing through the point $s = 1$. This function is related to the partial sums of the exponential function by the formula

$$ F_n(z) = \begin{cases} -(ez)^{-n} p_{n-1}[\exp](nz) & \text{for } z \text{ outside } \gamma, \\ (ze^{1-z})^{-n} - (ez)^{-n} p_{n-1}[\exp](nz) & \text{for } z \neq 0 \text{ inside } \gamma. \end{cases} $$

Taking into account its decay as $|z| \to \infty$ and the above jump as z crosses the curve γ, the authors formulated a Riemann-Hilbert problem (see Subsection 3.2) solved by $F_n(z)$. By applying the Riemann-Hilbert method to this problem they obtained a complete asymptotic expansion for the scaled partial sums $p_{n-1}[\exp](nz)$ as $n \to \infty$ which is valid for all z in a neighborhood of the point $z = 1$, thereby considerably improving on Newman and Rivlin's limit in Theorem 3. From this result the authors were able to obtain complete asymptotic expansions of the zeros of the scaled partial sums $p_{n-1}[\exp](nz)$ which approach the point $z = 1$.

The methods in this paper, and in particular in Section 6, are based on the ones Kriecherbauer, Kuijlaars, Mclaughlin, and Miller used to obtain the above results.

1.2 This paper

In this paper we will verify the Modified Saff-Varga Width Conjecture for a class of entire functions of positive, finite order with a certain asymptotic character in their sectors of maximal growth.

We will focus on functions f which are "exponential-like" in the sense that $f(z) \approx \exp(z^\lambda)$ in one or more sectors in the plane, and which are bounded by a smaller exponential $\exp(\mu|z|^\lambda)$, $\mu < 1$, otherwise. For the sake of generality we will allow f to have some subexponential factors in its asymptotic—in the case of a single sector of maximal growth we will assume that $f(z) \sim z^a (\log z)^b \exp(z^\lambda)$ for some $a, b \in \mathbb{C}$, and in the case of two or more sectors of maximal growth we will restrict this a little and only assume that $f(z) \sim z^a \exp(z^\lambda)$ in those sectors. Precise statements of these assumptions are given in Section 2.

Remark 1 These asymptotic assumptions can be generalized to include other reasonably simple subexponential factors such as $(\log \log z)^c$. For the sake of simplifying the exposition we will stick to simpler cases.

We will begin in Section 4 by deriving the limit curves for the appropriately-scaled zeros of the partial sums of these functions f in a maximal exponential growth sector $|\arg z| < \theta$. Then in Section 5 we will prove analogues of Theorem 2 which we refer to as "scaling limits at the arcs of the limit curve". We will then use these scaling limits to verify part (a) of the Modified Saff-Varga Width Conjecture for these functions f in the sector $0 < |\arg z| < \theta$. Finally in Section 6 we will prove analogues of Theorems 1, 3, and 4 which we refer to as "scaling limits at the corner of the limit curve". We will use these scaling limits to verify part (b) of the Modified Saff-Varga Width Conjecture at the exceptional argument $\arg z = 0$.

In total we verify the Modified Saff-Varga Width Conjecture for these functions f in the full maximal exponential growth sector $|\arg z| < \theta$.

In Section 7 we will apply the results in the preceding sections to several common special functions. Among these will be the sine and cosine functions [23, 9, 24, 25, 26], the confluent hypergeometric functions [15], the Bessel functions of the first kind [27], and certain exponential integrals [16, 27], all of which have been studied before in some way in the listed citations (though some in less generality, e.g. with tighter restrictions on the ranges of their parameters). For these functions, scaling limits of the form in this paper have only been obtained for the confluent hypergeometric functions (see [15]). We will also consider the Airy functions and the parabolic cylinder functions, neither of which have, to my knowledge, been previously examined in this context.

Remark 2 Varga and Carpenter state in [25] that the study of the behavior of the zeros of the partial sums of sine and cosine near the top and bottom corners of their limit curves is an open problem. We solve this problem in Subsection 7.1 of this paper by applying Theorem 14. See, for example, the approximations obtained in Figure 4.

2 Definitions

2.1 Functions with one direction of maximal exponential growth

Let $a, b \in \mathbb{C}$, $0 < \lambda < \infty$, $0 < \theta < \pi$, and $\mu < 1$. We say that an entire function f has one direction of maximal exponential growth if

$$f(z) = \begin{cases} z^a (\log z)^b \exp\bigl(z^\lambda\bigr) \left[1 + o(1)\right] & \text{for } |\arg z| \le \theta, \\ O\bigl(\exp(\mu |z|^\lambda)\bigr) & \text{for } |\arg z| > \theta \end{cases} \tag{1}$$

as $|z| \to \infty$, with each estimate holding uniformly in its sector.

Note that without loss of generality we assume that the sector of maximal growth is symmetric about the positive real line—if a function grows maximally in some other direction we can replace z by ζz for some appropriate $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ so that the maximal growth sector of the rotated function is oriented as desired. Similarly note that we assume f has been normalized so that the leading coefficient in its asymptotic, as well as the coefficient of z^λ in the exponential, are both equal to 1.

For such an f, let

$$p_n(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k$$

denote the n^{th} partial sum of its power series and define

$$r_n = \left(\frac{n}{\lambda}\right)^{1/\lambda}. \tag{2}$$

For $|\arg z| \le \theta$ it follows from (1) that

$$\begin{aligned} \frac{f(r_n z)}{r_n^a (\log r_n)^b (e^{1/\lambda} z)^n} &\sim z^a \left(z^\lambda e^{1 - z^\lambda}\right)^{-n/\lambda} \\ &= z^a e^{n\varphi(z)} \end{aligned} \tag{3}$$

as $n \to \infty$, where

$$\varphi(z) := \left(z^\lambda - 1 - \lambda \log z\right) \Big/ \lambda. \tag{4}$$

A straightforward calculation shows that $\varphi(1) = \varphi'(1) = 0$ and $\varphi''(1) = \lambda$, so

$$\varphi(s) = \frac{\lambda}{2}(s-1)^2 + O\Bigl((s-1)^3\Bigr)$$

in a neighborhood of $s = 1$. The inverse function theorem ensures the existence of a neighborhood V of the origin, a neighborhood $U \subset U_\gamma$ of $s = 1$, and a biholomorphic map $\psi \colon V \to U$ which satisfies

$$(\varphi \circ \psi)(x) = x^2 \tag{5}$$

for $x \in V$. This function ψ maps a segment of the imaginary axis onto the path of steepest descent of the function $\operatorname{Re} \varphi(z)$ going through $z = 1$ with $\psi(0) = 1$, and we make the choice that $\psi'(0) = \sqrt{2/\lambda}$.

Definition 1 A contour γ is said to be *admissible* for φ if

1. γ is a smooth Jordan curve winding counterclockwise around the origin.
2. In the sector $|\arg z| \le \theta$, γ is a positive distance from the curve $\operatorname{Re}\varphi(z) = 0$ except for a part that lies in some neighborhood U_γ of $z = 1$. In this set U_γ the contour γ coincides with the path of steepest decent of the function $\operatorname{Re}\varphi(z)$ passing through the point $z = 1$.
3. In the sector $|\arg z| \ge \theta$, γ coincides with the unit circle.

Let γ be an admissible contour for φ and suppose for now that $z \neq 0$ is inside the scaled contour $r_n\gamma$. The function

$$\frac{f(z) - p_{n-1}(z)}{z^n} =: \Phi(z)$$

is entire, so by Cauchy's integral formula

$$\Phi(z) = \frac{1}{2\pi i}\int_{r_n\gamma} \zeta^{-n} f(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{r_n\gamma} \zeta^{-n} p_{n-1}(\zeta)\frac{d\zeta}{\zeta - z}. \tag{6}$$

Since

$$\int_{r_n\gamma} \zeta^{-m}\frac{d\zeta}{\zeta - z} = 0$$

for all integers $m \ge 1$, the second integral in (6) is zero. Making the substitution $\zeta = r_n s$ and replacing z by $r_n z$ yields the identity

$$\frac{f(r_n z) - p_{n-1}(r_n z)}{(r_n z)^n} = \frac{1}{2\pi i}\int_\gamma (r_n s)^{-n} f(r_n s)\frac{ds}{s - z},$$

which holds for $z \neq 0$ inside γ. (This construction is a special case of the one in [3, p. 436] for an integral representation of the error of a Padé approximation.)

Define the function

$$F_n(z) = \frac{r_n^{-a}(\log r_n)^{-b}}{2\pi i}\int_\gamma \left(e^{1/\lambda} s\right)^{-n} f(r_n s)\frac{ds}{s - z} \tag{7}$$

for $z \notin \gamma$, $z \neq 0$. For z inside γ with $z \neq 0$ it follows from the derivation above that

$$F_n(z) = \frac{f(r_n z) - p_{n-1}(r_n z)}{r_n^a (\log r_n)^b (e^{1/\lambda} z)^n}.$$

The value of $F_n(z)$ for z outside γ can be calculated using a similar derivation or by using the residue theorem, and in total

$$F_n(z) = \frac{1}{r_n^a (\log r_n)^b (e^{1/\lambda} z)^n} \times \begin{cases} -p_{n-1}(r_n z) & \text{for } z \text{ outside } \gamma, \\ f(r_n z) - p_{n-1}(r_n z) & \text{for } z \neq 0 \text{ inside } \gamma. \end{cases} \tag{8}$$

In particular we have

$$F_n^+(z) = F_n^-(z) + \frac{f(r_n z)}{r_n^a (\log r_n)^b (e^{1/\lambda} z)^n}, \qquad z \in \gamma,$$

where F_n^+ (resp. F_n^-) refers to the continuous extension of F_n from inside (resp. outside) γ onto γ.

Let $\gamma_\theta = \gamma \cap \{z \in \mathbb{C} : |\arg z| \le \theta\}$ and define

$$G_n(z) = \frac{1}{2\pi i} \int_{\gamma_\theta} e^{n\varphi(s)} \frac{ds}{s-z}, \tag{9}$$

where φ is as in (4). Plemelj's formula (Proposition 2) implies that

$$G_n^+(z) = G_n^-(z) + e^{n\varphi(z)}, \qquad z \in \gamma_\theta,$$

where G_n^+ and G_n^- refer to the continuous extensions of G_n from the left and right of γ_θ onto γ_θ, respectively. Based on the asymptotic (3) and the fact that the saddle point of the function $\varphi(s)$ is located at $s = 1$, we expect that $F_n(z) \approx G_n(z)$ for $z \approx 1$ as $n \to \infty$. Something to this effect is shown in Lemma 7.

Just as in [10, p. 189] we define

$$h(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{u-\zeta}, \qquad \zeta \in \mathbb{C} \setminus \mathbb{R}$$

and

$$P_n(z) = h\Big(-i\sqrt{n}\psi^{-1}(z)\Big), \qquad z \in U \setminus \gamma_\theta, \tag{10}$$

where ψ and U are as in (5). Plemelj's formula implies that

$$h^+(x) = h^-(x) + e^{-x^2}, \qquad x \in \mathbb{R},$$

where $\mathbb{R}$ is given the usual orientation from $-\infty$ to $+\infty$, and setting $z = \psi(ix/\sqrt{n})$ yields

$$P_n^+(z) = P_n^-(z) + e^{n\varphi(z)}, \qquad z \in U \cap \gamma_\theta.$$

Here $+$ and $-$ indicate approaching the contour γ_θ from the left and from the right, respectively.

2.2 Functions with two directions of maximal exponential growth

Let $a, b, A \in \mathbb{C}$, $0 < \lambda < \infty$, $\mu < 1$, and $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, $\zeta \neq 1$. Let $\theta \in (0, \pi)$ be small enough so that the sectors $|\arg z| \le \theta$ and $|\arg(z/\zeta)| \le \theta$ are disjoint. We say that an entire function f has two directions of maximal exponential growth if

$$f(z) = \begin{cases} z^a \exp\big(z^\lambda\big)\,\big[1 + o(1)\big] & \text{for } |\arg z| \le \theta, \\ A(z/\zeta)^b \exp\big((z/\zeta)^\lambda\big)\,\big[1 + o(1)\big] & \text{for } |\arg(z/\zeta)| \le \theta, \\ O\big(\exp\big(\mu|z|^\lambda\big)\big) & \text{otherwise} \end{cases} \tag{11}$$

as $|z| \to \infty$, with each estimate holding uniformly in its sector.

Without loss of generality we assume that one sector of maximal growth is symmetric about the positive real line—if neither of the function's directions of maximal growth are bisected by the positive real line then we can replace z by ωz for some $\omega \in \mathbb{C}$ with $|\omega| = 1$ so that one of those directions is as desired. Similarly we assume that f has been normalized so that the leading coefficient, as well as the coefficient of z^λ in the first exponential and the coefficient of $(z/\zeta)^\lambda$ in the second exponential, are all equal to 1 in its asymptotic along the positive real line.

For this f, let

$$p_n(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k,$$

define

$$r_n = \left(\frac{n}{\lambda}\right)^{1/\lambda}, \tag{12}$$

let

$$\varphi(z) = \left(z^\lambda - 1 - \lambda \log z\right) \Big/ z,$$

and let γ be an admissible contour for φ, just as in Subsection 2.1.

Define

$$F_n(z) = \frac{r_n^{-a}}{2\pi i} \int_\gamma \left(e^{1/\lambda} s\right)^{-n} f(r_n s) \frac{ds}{s-z} \tag{13}$$

for $z \notin \gamma$, $z \neq 0$. Analogous to the computation in subection 2.1,

$$F_n(z) = \frac{1}{r_n^a (e^{1/\lambda} z)^n} \times \begin{cases} -p_{n-1}(r_n z) & \text{for } z \text{ outside } \gamma, \\ f(r_n z) - p_{n-1}(r_n z) & \text{for } z \neq 0 \text{ inside } \gamma. \end{cases} \tag{14}$$

2.3 Functions with more directions of maximal exponential growth

Let $a, b_1, \ldots, b_m, A_1, \ldots, A_m \in \mathbb{C}$, $0 < \lambda < \infty$, $\mu < 1$, and $\zeta_1, \ldots, \zeta_m \in \mathbb{C}$ with $|\zeta_k| = 1$, $\zeta_k \neq 1$ for all $k = 1, \ldots, m$ and $\zeta_j \neq \zeta_k$ for $j \neq k$. Let $\theta \in (0, \pi)$ be small enough so that all of the sectors $|\arg z| \leq \theta$, $|\arg(z/\zeta_k)| \leq \theta$, $k = 1, \ldots, m$, are disjoint. We say that an entire function f is a has $m + 1$ directions of maximal exponential growth if

$$f(z) = \begin{cases} z^a \exp\left(z^\lambda\right) \left[1 + o(1)\right] & \text{for } |\arg z| \leq \theta, \\ A_1 (z/\zeta_1)^{b_1} \exp\left((z/\zeta_1)^\lambda\right) \left[1 + o(1)\right] & \text{for } |\arg(z/\zeta_1)| \leq \theta, \\ \quad \vdots & \\ A_m (z/\zeta_m)^{b_m} \exp\left((z/\zeta_m)^\lambda\right) \left[1 + o(1)\right] & \text{for } |\arg(z/\zeta_m)| \leq \theta, \\ O\left(\exp\left(\mu |z|^\lambda\right)\right) & \text{otherwise} \end{cases} \tag{15}$$

as $|z| \to \infty$, with each estimate holding uniformly in its sector. For this f, let, $p_n(z)$, r_n, and $F_n(z)$ be defined as in Subsection 2.2. For convenience of notation, define $b_0 = a$, $A_0 = 1$, and $\zeta_0 = 1$.

3 Technical preliminaries

3.1 Cauchy integrals

Here we will state two important facts about Cauchy integrals. Proofs of these as well as further details can be found in [12, 13, 6].

Definition 2 (Cauchy integrals) Let L be a finite, smooth, oriented curve (which may be a closed contour) and suppose that $g\colon \mathbb{C} \to \mathbb{C}$ is integrable with respect to arc length on L, i.e.

$$\int_L |g(t)|\,|dt| < \infty.$$

For $z \in \mathbb{C} \setminus L$ the Cauchy integral of g is defined as

$$\mathcal{C}_L[g](z) = \frac{1}{2\pi i} \int_L \frac{g(t)}{t-z}\,dt.$$

Proposition 1 *The function $\mathcal{C}_L[g]$ is analytic on $\mathbb{C} \setminus L$ and $\mathcal{C}_L[g](z) = O(z^{-1})$ as $|z| \to \infty$.*

Proposition 2 (Plemelj formulas) *If g is Hölder continuous on L then $\mathcal{C}_L[g]$ has extensions from the left of L onto L and from the right of L onto L denoted by $\mathcal{C}_L^+[g]$ and $\mathcal{C}_L^-[g]$, respectively, which are continuous except possibly in arbitrarily small neighborhoods of the endpoints of L. If $\zeta \in L$ is not an endpoint of L, or if it is an endpoint and $g(\zeta) = 0$, then*

$$\mathcal{C}_L^\pm[g](\zeta) = \frac{1}{2\pi i}\,\mathrm{P.\,V.}\int_L \frac{g(t)}{t-\zeta}\,dt \pm \frac{g(\zeta)}{2},$$

where $\mathrm{P.\,V.}\int$ is a principal value integral, and hence

$$\mathcal{C}_L^+[g](\zeta) - \mathcal{C}_L^-[g](\zeta) = g(\zeta).$$

If L is an arc which connects a to b, $a, b \in \mathbb{C}$, then there exists a function H_a defined in a neighborhood of a which is analytic on L^c (the complement of L), continuous at a, and has continuous extensions onto $L \setminus \{a\}$ from the left and the right such that

$$\mathcal{C}_L[g](z) = \frac{g(a)}{2\pi i} \log \frac{1}{z-a} + H_a(z)$$

for z near a, where the branch cut of the logarithm coincides with L. For z near b there is an analogous function H_b such that

$$\mathcal{C}_L[g](z) = -\frac{g(b)}{2\pi i} \log \frac{1}{z-b} + H_b(z),$$

where again the branch cut of the logarithm coincides with L.

3.2 Scalar Riemann-Hilbert problems

Let L be an oriented curve in the plane and let g be a function defined on L. If L is a closed contour set $L^* = L$, and if L is an arc which begins at $z = a$ and ends at $z = b$, set $L^* = L \setminus \{a, b\}$. A scalar Riemann-Hilbert problem is a problem of the following form.

Riemann-Hilbert Problem 1. *Seek an analytic function $\Phi\colon \mathbb{C} \setminus L \to \mathbb{C}$ such that*

1. *Φ has continuous extensions Φ^+ and Φ^- from the left and right of L^*, respectively, onto L^*,*
2. *$\Phi^+(\zeta) - \Phi^-(\zeta) = g(\zeta)$ for $\zeta \in L^*$,*
3. *$\Phi(z) \to 0$ as $|z| \to \infty$,*
4. *if c is an endpoint of L then $\Phi(z) = O(|z-c|^{-1+\epsilon})$ as $z \to c$ with $z \in \mathbb{C} \setminus L$ for some $\epsilon > 0$.*

Proposition 3 *If L has finite length and if g is Hölder continuous on L then $\Phi = \mathcal{C}_L[g]$ is the unique solution to Riemann Hilbert Problem 1.*

A proof of this result can be found in [13, sec. 78].

4 Limit curves for the zeros of the partial sums

In this section we will calculate the set of limit points in a certain sector of the (appropriately scaled) zeros of the partial sums of the Maclaurin series for the functions f we are interested in. Just as in the examples discussed in Section 1, as the degree of the partial sum tends to infinity its scaled zeros converge to a piecewise-smooth curve in the plane.

4.1 One direction of maximal exponential growth

Using the definitions in Subsection 2.1, let f be an entire function with a single direction of maximal exponential growth.

Theorem 5 *All limit points of the zeros of the scaled partial sums $p_{n-1}(r_n z)$ in the sector $|\arg z| < \theta$, $z \neq 0$ lie on the curve*

$$\left| z^\lambda \exp\left(1 - z^\lambda\right) \right| = 1, \qquad |z| \leq 1.$$

Further, these zeros approach this curve from the region $|z^\lambda \exp(1-z^\lambda)| > 1$.

In fact we will show that if $z = z(n)$ is a zero of $p_{n-1}(r_n z)$ which converges to a point z_0 with $|\arg z_0| < \theta$, $z_0 \neq 0, 1$ then

$$\left| z^\lambda \exp\left(1 - z^\lambda\right) \right| = 1 + \frac{\lambda \log n}{2n} + O\left(n^{-1}\right)$$

as $n \to \infty$.

4.1.1 Proof of Theorem 5

Split the integral for F_n into the two pieces

$$F_n(z) = \frac{r_n^{-a} (\log r_n)^{-b}}{2\pi i} \left(\int_{\gamma_\theta} + \int_{\gamma \setminus \gamma_\theta} \right) \left(e^{1/\lambda} s \right)^{-n} f(r_n s) \frac{ds}{s - z}, \tag{16}$$

where γ_θ is the portion of γ in the sector $|\arg z| \leq \theta$.

Lemma 1 *If z is restricted to any sector $|\arg z| \le \theta - \epsilon$ with $\epsilon > 0$ then*

$$\int_{\gamma \setminus \gamma_\theta} \left(e^{1/\lambda} s\right)^{-n} f(r_n s) \frac{ds}{s-z} = O\left(e^{(\mu-1)n/\lambda}\right)$$

as $n \to \infty$ uniformly in z.

Proof. For any $\epsilon > 0$ there exists a constant $K > 0$ such that $|s - z| \ge K$ for all $s \in \gamma \setminus \gamma_\theta$ and all z with $|\arg z| \le \theta - \epsilon$. Also, it follows from the asymptotic assumption on f in (1) that for all z large enough with $|\arg z| \ge \theta$ there is a constant M such that

$$|f(z)| \le M \exp\left(\mu |z|^\lambda\right).$$

Since the contour $\gamma \setminus \gamma_\theta$ lies on the unit circle in the sector $|\arg z| \ge \theta$, it therefore follows from the above assumptions that

$$\left| \int_{\gamma \setminus \gamma_\theta} \left(e^{1/\lambda} s\right)^{-n} f(r_n s) \frac{ds}{s-z} \right| \le K^{-1} M \operatorname{length}(\gamma \setminus \gamma_\theta) e^{(\mu-1)n/\lambda}$$

for all n large enough. □

Define the function $\delta(z)$ for $|\arg z| \le \theta$ by

$$f(z) = z^a (\log z)^b \exp(z^\lambda) \big[1 + \delta(z)\big]. \tag{17}$$

This implies

$$\frac{f(r_n s)}{r_n^a (\log r_n)^b (e^{1/\lambda} s)^n} = s^a e^{n\varphi(s)} \left(1 + \frac{\log s}{\log r_n}\right)^b \big[1 + \delta(r_n s)\big]$$

for $s \in \gamma_\theta$. By then defining

$$\tilde{\delta}(r_n, s) = \left(1 + \frac{\log s}{\log r_n}\right)^b \big[1 + \delta(r_n s)\big] - 1 \tag{18}$$

the integrand of the first integral in (16) can be rewritten as

$$s^a e^{n\varphi(s)} + s^a e^{n\varphi(s)} \tilde{\delta}(r_n, s).$$

It then follows from (16) and Lemma 1 that, for some constant $c > 0$,

$$F_n(z) = \frac{1}{2\pi i} \int_{\gamma_\theta} s^a e^{n\varphi(s)} \frac{ds}{s-z} + \frac{1}{2\pi i} \int_{\gamma_\theta} s^a e^{n\varphi(s)} \tilde{\delta}(r_n, s) \frac{ds}{s-z} + O(e^{-cn}) \tag{19}$$

as $n \to \infty$ uniformly for z restricted to any sector $|\arg z| \le \theta - \epsilon$ with $\epsilon > 0$.

For $\epsilon > 0$ define N_ϵ to be the set of all points within a distance of ϵ of the curve γ_θ.

Lemma 2

$$\int_{\gamma_\theta} s^a e^{n\varphi(s)} \frac{ds}{s-z} = \frac{i}{1-z} \sqrt{\frac{2\pi}{\lambda n}} + O\left(n^{-1}\right)$$

as $n \to \infty$ uniformly for $z \in \mathbb{C} \setminus N_\epsilon$ with $\epsilon > 0$.

Proof. Fix $\epsilon > 0$. Using the same method as in the proof of Lemma 1 it can be shown that the integral over the part of the contour outside of U is exponentially small, so that

$$\int_{\gamma_\theta} s^a e^{n\varphi(s)} \frac{ds}{s-z} = \int_{\gamma_\theta \cap U} s^a e^{n\varphi(s)} \frac{ds}{s-z} + O(e^{-cn})$$

as $n \to \infty$ uniformly for $z \in \mathbb{C}\backslash N_\epsilon$, where c is some positive constant not depending on n or z. Making the substitution $s = \psi(it)$ yields

$$\int_{\gamma_\theta} s^a e^{n\varphi(s)} \frac{ds}{s-z} = \int_{-\alpha_1}^{\alpha_2} e^{-nt^2} \psi(it)^a \frac{i\psi'(it)}{\psi(it)-z}\, dt + O(e^{-cn}) \tag{20}$$

for some $\alpha_1, \alpha_2 > 0$.

Now write

$$\psi(it)^a \frac{i\psi'(it)}{\psi(it)-z} = \frac{i}{1-z}\sqrt{\frac{2}{\lambda}} + \tilde{\psi}(t,z),$$

where

$$\tilde{\psi}(t,z) := \psi(it)^a \frac{i\psi'(it)}{\psi(it)-z} - \psi(0)^a \frac{i\psi'(0)}{\psi(0)-z}.$$

By taking U smaller if necessary it is tedious though straightforward to show by Taylor's theorem that for all $z \in \mathbb{C} \setminus N_\epsilon$ and for all $t \in [-\alpha_1, \alpha_2]$ that

$$|\tilde{\psi}(t,z)| \leq M|t|$$

for some constant M not depending on z, and therefore that

$$\begin{aligned}
\left| \int_{-\alpha_1}^{\alpha_2} e^{-nt^2} \tilde{\psi}(t,z)\, dt \right| &\leq M \int_{-\alpha_1}^{\alpha_2} e^{-nt^2} |t|\, dt \\
&< 2M \int_0^\infty e^{-nt^2} t\, dt \\
&= \frac{M}{n}.
\end{aligned}$$

Substituting this into (20) and using the fact that

$$\int_{-\alpha_1}^{\alpha_2} e^{-nt^2}\, dt = \sqrt{\frac{\pi}{n}} + O(e^{-c'n})$$

for some constant $c' > 0$ yields the estimate

$$\begin{aligned}
\int_{\gamma_\theta} s^a e^{n\varphi(s)} \frac{ds}{s-z} &= \frac{i}{1-z}\sqrt{\frac{2}{\lambda}} \int_{-\alpha_1}^{\alpha_2} e^{-nt^2}\, dt + O\Big(n^{-1}\Big) \\
&= \frac{i}{1-z}\sqrt{\frac{2\pi}{\lambda n}} + O\Big(n^{-1}\Big)
\end{aligned} \tag{21}$$

as $n \to \infty$ uniformly for $z \in \mathbb{C} \setminus N_\epsilon$. □

Lemma 3

$$\int_{\gamma_\theta} s^a e^{n\varphi(s)} \tilde{\delta}(r_n, s) \frac{ds}{s-z} = o\Big(n^{-1/2}\Big)$$

as $n \to \infty$ uniformly for $z \in \mathbb{C} \setminus N_\epsilon$ with $\epsilon > 0$.

Proof. Fix $\epsilon > 0$ and let $U' \subset U$ be a neighborhood of $z = 1$ such that

$$\sup_{z \in U'} |z - 1| < \epsilon.$$

As in Lemma 3,

$$\int_{\gamma_\theta} s^a e^{n\varphi(s)} \tilde{\delta}(r_n, s) \frac{ds}{s - z} = \int_{\gamma_\theta \cap U'} s^a e^{n\varphi(s)} \tilde{\delta}(r_n, s) \frac{ds}{s - z} + O(e^{-cn})$$

as $n \to \infty$ uniformly for $z \in \mathbb{C} \backslash N_\epsilon$, where c is some positive constant not depending on n or z, and the substitution $s = \psi(it)$ yields

$$\int_{\gamma_\theta} s^a e^{n\varphi(s)} \tilde{\delta}(r_n, s) \frac{ds}{s - z} = \int_{-\alpha_1'}^{\alpha_2'} e^{-nt^2} \psi(it)^a \frac{i\psi'(it)}{\psi(it) - z} \tilde{\delta}(r_n, \psi(it))\, dt + O(e^{-cn})$$

for some $\alpha_1', \alpha_2' > 0$. If $z \in \mathbb{C} \setminus N_\epsilon$ and $t \in [-\alpha_1', \alpha_2']$ then $|\psi(it) - z| \geq K$ for some positive constant K, so that

$$\begin{aligned} &\left| \int_{-\alpha_1'}^{\alpha_2'} e^{-nt^2} \psi(it)^a \frac{i\psi'(it)}{\psi(it) - z} \delta(n\psi(it))\, dt \right| \\ &\quad \leq K^{-1} \int_{-\alpha_1'}^{\alpha_2'} e^{-nt^2}\, dt \sup_{-\alpha_1' < t < \alpha_2'} \left| \psi(it)^a \psi'(it) \tilde{\delta}(r_n, \psi(it)) \right| \\ &\quad < \frac{M}{\sqrt{n}} \sup_{-\alpha_1' < t < \alpha_2'} \left| \tilde{\delta}(r_n, \psi(it)) \right| \end{aligned}$$

for some positive constant M. Note that U may need to be made smaller to ensure that $\psi'(it)$ is bounded—doing this doesn't cause any issues.

By definition $\delta(z) \to 0$ as $|z| \to \infty$ uniformly for $|\arg z| \leq \theta$, and so $\delta(r_n s) \to 0$ as $n \to \infty$ uniformly for $s \in \gamma_\theta$. By extension this holds for $\tilde{\delta}(r_n, s)$ as well, and thus

$$\lim_{n \to \infty} \sup_{-\alpha_1' < t < \alpha_2'} \left| \tilde{\delta}(r_n, \psi(it)) \right| = 0.$$

□

Combining Lemmas 2 and 3 and equation (19) yields the asymptotic

$$F_n(z) = \frac{1}{(1 - z)\sqrt{2\pi\lambda n}} + o\left(n^{-1/2}\right) \tag{22}$$

as $n \to \infty$ uniformly for $z \in \mathbb{C} \setminus N_\epsilon$ with $|\arg z| \leq \theta - \epsilon$ for any fixed $\epsilon > 0$.

Suppose that $z = z(n)$ is a zero of $p_{n-1}(r_n z)$, i.e. that $p_{n-1}(r_n z) = 0$. Suppose further that, as $n \to \infty$, z tends to a limit point inside γ and in the set

$$\{z \in \mathbb{C} : |\arg z| < \theta \text{ and } z \neq 0, 1\}.$$

Then for n large enough there is an $\epsilon > 0$ such that $|\arg z| \leq \theta - \epsilon$ and $z \in \mathbb{C} \setminus N_\epsilon$, and so from the definition of $F_n(z)$ it follows from (22) that

$$\frac{f(r_n z)}{r_n^a (\log r_n)^b (e^{1/\lambda} z)^n} \sim \frac{1}{(1 - z)\sqrt{2\pi\lambda n}} = \frac{e^{-(\log n)/2}}{(1 - z)\sqrt{2\pi\lambda}}$$

as $n \to \infty$, and from the asymptotic assumption on f in (1) that

$$z^a \left[z^\lambda \exp\left(1 - z^\lambda\right)\right]^{-n/\lambda} \sim \frac{e^{-(\log n)/2}}{(1-z)\sqrt{2\pi\lambda}} \tag{23}$$

as $n \to \infty$. If $g(n) = \Theta(1)$ then

$$|g(z)|^{1/n} = 1 + O\left(n^{-1}\right)$$

as $n \to \infty$, so taking absolute values and raising both sides of (23) to the power $-\lambda/n$ yields

$$\begin{aligned} \left|z^\lambda \exp\left(1 - z^\lambda\right)\right| &= \exp\left(\frac{\lambda \log n}{2n}\right)\left[1 + O\left(n^{-1}\right)\right] \\ &= 1 + \frac{\lambda \log n}{2n} + O\left(n^{-1}\right) \end{aligned} \tag{24}$$

as $n \to \infty$.

It follows from the above asymptotic that the limit points z_0 of the zeros of $p_{n-1}(r_n z)$ inside γ with $|\arg z_0| < \theta$, $z_0 \neq 0$ lie on the curve

$$\left|z_0^\lambda \exp\left(1 - z_0^\lambda\right)\right| = 1,$$

or, equivalently,

$$\operatorname{Re} \varphi(z_0) = 0.$$

Further, the exterior of this curve in this region is characterized by the inequality

$$\left|z^\lambda \exp\left(1 - z^\lambda\right)\right| > 1,$$

so the zeros approach this limit curve from the exterior. Finally, as an admissible contour may be taken to lie as close to the lines $\operatorname{Im} \varphi(z) = 0$ as desired, the only such limit points z_0 in the whole set

$$\{z \in \mathbb{C} : \operatorname{Re} \varphi(z) > 0\} \cup \{z \in \mathbb{C} : \operatorname{Re} \varphi(z) \leq 0 \text{ and } |z| \leq 1\}$$

must lie on the curve $\operatorname{Re} \varphi(z_0) = 0$.

Suppose now that z is a zero of $p_{n-1}(r_n z)$ which lies in a sector $|\arg z| \leq \theta - \epsilon$ with $\epsilon > 0$ and in the set

$$\{z \in \mathbb{C} : \operatorname{Re} \varphi(z) \leq 0 \text{ and } |z| > 1\} \setminus N_\epsilon$$

for n large enough. Then z eventually lies outside of γ, and so $F_n(z) = 0$ by (8). From equation (22) it follows that

$$0 = \frac{1}{(1-z)\sqrt{2\pi\lambda n}} + o\left(n^{-1/2}\right)$$

as $n \to \infty$, and, on multiplying this by $\sqrt{n}$, that

$$0 = \frac{1}{(1-z)\sqrt{2\pi\lambda}} + o(1)$$

as $n \to \infty$. The only way this is possible is if $|z| \to \infty$.

Since $\epsilon > 0$ was arbitrary, it follows that the zeros of $p_{n-1}(r_n z)$ have no limit point in the set

$$\{z \in \mathbb{C} : \operatorname{Re} \varphi(z) \leq 0 \text{ and } |z| > 1\}.$$

This completes the proof.

4.2 Two directions of maximal exponential growth

Using the definitions in Subsection 2.2, let f be an entire function with two directions of maximal exponential growth.

Theorem 6 *All limit points of the zeros of the scaled partial sums $p_{n-1}(r_n z)$ in the sector $|\arg z| < \theta$, $z \neq 0$ lie on the curve*

$$\left| z^\lambda \exp\left(1 - z^\lambda\right) \right| = 1, \qquad |z| \leq 1.$$

If $\operatorname{Re} a \neq \operatorname{Re} b$, define $\alpha = \min\{\operatorname{Re} a, \operatorname{Re} b\}$. In this case the zeros approach this limit curve from the region $|z^\lambda \exp(1 - z^\lambda)| > 1$ if $\alpha - \operatorname{Re} b + \lambda/2 > 0$ and from the region $|z^\lambda \exp(1 - z^\lambda)| < 1$ if $\alpha - \operatorname{Re} b + \lambda/2 < 0$.

In fact we will show that if $\operatorname{Re} a \neq \operatorname{Re} b$ and if $z = z(n)$ is a zero of $p_{n-1}(r_n z)$ which converges to a point z_0 with $|\arg z_0| < \theta$, $z_0 \neq 0$ then

$$\left| z^\lambda \exp\left(1 - z^\lambda\right) \right| = 1 + \left(\alpha - \operatorname{Re} b + \frac{\lambda}{2}\right) \frac{\log n}{n} + O\left(n^{-1}\right)$$

as $n \to \infty$. This formula also holds when $\operatorname{Re} a = \operatorname{Re} b$ as long as

$$z_0 \notin \{w \in \mathbb{C} : |\zeta - w| = |A(1 - w)|\}.$$

4.2.1 Proof of Theorem 6

Call γ_1 the part of γ in the sector $|\arg z| \leq \theta$, call γ_2 the part of γ in the sector $|\arg(z/\zeta)| \leq \theta$, and call γ_3 the part of γ outside of either of those sectors. This allows us to divide the integral in the definition of $F_n(z)$ in (13) into three parts

$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}$$

which will be analyzed separately.

Using a method identical to the proof of Lemma 1 it can be shown that

$$\int_{\gamma_3} \left(e^{1/\lambda} s\right)^{-n} f(r_n s) \frac{ds}{s - z} = O\left(e^{(\mu - 1)n/\lambda}\right) \tag{25}$$

as $n \to \infty$ uniformly for z restricted to any sector $|\arg z| \leq \theta - \epsilon$ with $\epsilon > 0$, and using a method identical to the proofs of Lemmas 2 and 3 that

$$\int_{\gamma_1} \left(e^{1/\lambda} s\right)^{-n} f(r_n s) \frac{ds}{s - z} = \frac{i r_n^a}{1 - z} \sqrt{\frac{2\pi}{\lambda n}} + o\left(r_n^a n^{-1/2}\right) \tag{26}$$

as $n \to \infty$ uniformly for z in any set $\mathbb{C} \setminus N_\epsilon$ with $\epsilon > 0$. Here N_ϵ is defined to be the set of all points within a distance of ϵ from γ_1.

For $\epsilon > 0$ define $\tilde{N}_\epsilon$ to be the set of all points within a distance of ϵ of the curve γ_2.

Lemma 4

$$\int_{\gamma_2} \left(e^{1/\lambda}s\right)^{-n} f(r_n s)\frac{ds}{s-z} = \frac{iA\zeta^{1-n}r_n^b}{\zeta - z}\sqrt{\frac{2\pi}{\lambda n}} + o\left(r_n^b n^{-1/2}\right)$$

as $n \to \infty$ uniformly for $z \in \mathbb{C} \setminus \tilde{N}_\epsilon$ with $\epsilon > 0$.

Proof. By the asymptotic assumption on f in (11), for $|\arg(z/\zeta)| \le \theta$ we can write

$$f(z) = A(z/\zeta)^b \exp\left((z/\zeta)^\lambda\right)\left[1 + \delta(z)\right], \tag{27}$$

where $\delta(z) \to 0$ uniformly as $|z| \to \infty$ in this sector. This implies

$$\frac{f(r_n s)}{(e^{1/\lambda}s)^n} = A\zeta^{-n}r_n^b(s/\zeta)^b e^{n\tilde{\varphi}(s)}\left[1 + \delta(r_n s)\right]$$

for $s \in \gamma_2$, where

$$\tilde{\varphi}(s) = \left[(s/\zeta)^\lambda - 1 - \lambda\log(s/\zeta)\right]\Big/\lambda,$$

allowing us to split the integral in question like

$$\begin{aligned}\int_{\gamma_2} &\left(e^{1/\lambda}s\right)^{-n} f(r_n s)\frac{ds}{s-z} \\ &= A\zeta^{-n}r_n^b\left(\int_{\gamma_2}(s/\zeta)^b e^{n\tilde{\varphi}(s)}\frac{ds}{s-z} + \int_{\gamma_2}(s/\zeta)^b e^{n\tilde{\varphi}(s)}\delta(r_n s)\frac{ds}{s-z}\right).\end{aligned} \tag{28}$$

By the inverse function theorem there exists a neighborhood $\tilde{V}$ of the origin, a neighborhood $\tilde{U}$ of $s = \zeta$, and a biholomorphic map $\tilde{\psi}\colon \tilde{V} \to \tilde{U}$ which satisfies

$$(\tilde{\varphi} \circ \tilde{\psi})(x) = x^2$$

for $x \in \tilde{V}$. It follows that $\tilde{\psi}(0) = \zeta$, and we make the choice that $\tilde{\psi}'(0) = \zeta\sqrt{2/\lambda}$. This function $\tilde{\psi}$ maps a segment of the imaginary axis onto the path of steepest descent of the function $\operatorname{Re}\tilde{\varphi}(z)$ going through $z = \zeta$.

The rest of the proof proceeds just as the proofs of Lemmas 2 and 3 by using $\tilde{\varphi}$ and $\tilde{\psi}$ in place of φ and ψ, respectively. □

Combining equations (25), (26), and Lemma 4 yields the asymptotic

$$F_n(z) = \frac{1}{(1-z)\sqrt{2\pi\lambda n}} + \frac{A\zeta^{1-n}r_n^{b-a}}{(\zeta - z)\sqrt{2\pi\lambda n}} + o\left(n^{-1/2}\right) + o\left(r_n^{b-a}n^{-1/2}\right) \tag{29}$$

as $n \to \infty$ uniformly for $z \in \mathbb{C} \setminus N_\epsilon$ restricted to the sector $|\arg z| \le \theta - \epsilon$, where $\epsilon > 0$ is arbitrary but fixed.

Suppose that $z = z(n)$ is a zero of $p_{n-1}(r_n z)$ which, as $n \to \infty$, tends to a limit point inside γ and in the set

$$\{z \in \mathbb{C} : |\arg z| < \theta \text{ and } z \ne 0, 1\}.$$

Then for n large enough there is an $\epsilon > 0$ such that $|\arg z| \le \theta - \epsilon$ and $z \in \mathbb{C} \setminus N_\epsilon$, and so from the definition of $F_n(z)$ it follows from (29) that

$$\frac{f(r_n z)}{r_n^a(e^{1/\lambda}z)^n} = \frac{1}{(1-z)\sqrt{2\pi\lambda n}} + \frac{A\zeta^{1-n}r_n^{b-a}}{(\zeta - z)\sqrt{2\pi\lambda n}} + o\left(n^{-1/2}\right) + o\left(r_n^{b-a}n^{-1/2}\right)$$

as $n \to \infty$, and from the asymptotic assumption on f in (11) that

$$z^a \left[z^\lambda \exp\left(1 - z^\lambda\right)\right]^{-n/\lambda} \left[1 + o(1)\right]$$
$$= \frac{1}{(1-z)\sqrt{2\pi\lambda n}} + \frac{A\zeta^{1-n} r_n^{b-a}}{(\zeta - z)\sqrt{2\pi\lambda n}} + o\left(n^{-1/2}\right) + o\left(r_n^{b-a} n^{-1/2}\right) \tag{30}$$

as $n \to \infty$.

If $\operatorname{Re} a > \operatorname{Re} b$ then

$$z^a \left[z^\lambda \exp\left(1 - z^\lambda\right)\right]^{-n/\lambda} \sim \frac{1}{(1-z)\sqrt{2\pi\lambda n}}$$

and hence, just as in (24),

$$\left|z^\lambda \exp\left(1 - z^\lambda\right)\right| = 1 + \frac{\lambda \log n}{2n} + O\left(n^{-1}\right) \tag{31}$$

as $n \to \infty$. If instead $\operatorname{Re} a < \operatorname{Re} b$ then

$$z^a \left[z^\lambda \exp\left(1 - z^\lambda\right)\right]^{-n/\lambda} \sim \frac{A\zeta^{1-n} r_n^{b-a}}{(\zeta - z)\sqrt{2\pi\lambda n}}$$

and hence

$$\left|z^\lambda \exp\left(1 - z^\lambda\right)\right| = 1 + \left(\operatorname{Re} a - \operatorname{Re} b + \frac{\lambda}{2}\right) \frac{\log n}{n} + O\left(n^{-1}\right) \tag{32}$$

as $n \to \infty$. Finally suppose $\operatorname{Re} b = \operatorname{Re} a$, so that

$$z^a \left[z^\lambda \exp\left(1 - z^\lambda\right)\right]^{-n/\lambda} \left[1 + o(1)\right]$$
$$= \left(A^{-1} \frac{\zeta - z}{1 - z} + \zeta^{1-n} r_n^{b-a} + o(1)\right) \frac{1}{(\zeta - z)\sqrt{2\pi\lambda n}} \tag{33}$$

as $n \to \infty$. Note that $|\zeta^{1-n} r_n^{b-a}| = 1$ in this case, so for n large enough the quantity in parentheses is bounded below in absolute value by a positive constant unless

$$\left|\frac{\zeta - z}{1 - z}\right| \to |A|.$$

If z does not tend to a point on the circle

$$\{w \in \mathbb{C} : |\zeta - w| = |A(1 - w)|\}$$

then taking absolute values and raising both sides of the equation to the power $-\lambda/n$ yields

$$\left|z^\lambda \exp\left(1 - z^\lambda\right)\right| = 1 + \frac{\lambda \log n}{2n} + O\left(n^{-1}\right) \tag{34}$$

as $n \to \infty$. Suppose now that z does tend to a point on the circle

$$\{w \in \mathbb{C} : |\zeta - w| = |A(1 - w)|\},$$

define

$$M = \limsup_{n \to \infty} \left|A^{-1} \frac{\zeta - z}{1 - z} + \zeta^{1-n} r_n^{b-a}\right|,$$

and note that $M > 0$. It follows that

$$\liminf_{n\to\infty} \left| A^{-1}\frac{\zeta - z}{1-z} + \zeta^{1-n} r_n^{b-a} \right|^{-\lambda/n} = 1. \tag{35}$$

If

$$\limsup_{n\to\infty} \left| A^{-1}\frac{\zeta - z}{1-z} + \zeta^{1-n} r_n^{b-a} \right|^{-\lambda/n} \neq 1$$

then

$$\lim_{n\to\infty} \left| A^{-1}\frac{\zeta - z}{1-z} + \zeta^{1-n} r_n^{b-a} \right|^{-\lambda/n}$$

does not exist. However, by the above assumption on the convergence of z as $n \to \infty$ it's true that

$$\lim_{n\to\infty} \left| z^{-a\lambda/n} \right|, \ \lim_{n\to\infty} \left| z^{\lambda} \exp\Big(1 - z^{\lambda}\Big) \right|, \text{ and } \lim_{n\to\infty} \left| \frac{1}{(\zeta - z)\sqrt{2\pi\lambda n}} \right|^{-\lambda/n}$$

all exist. Consequently it follows from (33) that

$$\lim_{n\to\infty} \left| A^{-1}\frac{\zeta - z}{1-z} + \zeta^{1-n} r_n^{b-a} \right|^{-\lambda/n}$$

exists, and then from (35) that this limit is equal to 1. Altogether, this implies that

$$\left| z^{\lambda} \exp\Big(1 - z^{\lambda}\Big) \right| = 1 + o(1) \tag{36}$$

as $n \to \infty$.

It follows from the asymptotics in (31), (32), (34), and (36) that the limit points z_0 of the zeros of $p_{n-1}(r_n z)$ inside γ with $|\arg z_0| < \theta$, $z_0 \neq 0$ lie on the curve

$$\left| z_0^{\lambda} \exp\Big(1 - z_0^{\lambda}\Big) \right| = 1,$$

or, equivalently,

$$\operatorname{Re} \varphi(z_0) = 0.$$

Further, the exterior of this curve in this region is characterized by the inequality

$$\left| z^{\lambda} \exp\Big(1 - z^{\lambda}\Big) \right| > 1,$$

so if $\operatorname{Re} a \neq \operatorname{Re} b$ and we set $\alpha = \min\{\operatorname{Re} a, \operatorname{Re} b\}$ then the zeros approach this limit curve from the exterior if $\alpha - \operatorname{Re} b + \lambda/2 > 0$ and from the interior if $\alpha - \operatorname{Re} b + \lambda/2 < 0$. If $\operatorname{Re} a = \operatorname{Re} b$ then the zeros which approach points of the curve which are not on the circle

$$\{w \in \mathbb{C} : |\zeta - w| = |A(1-w)|\}$$

do so from the exterior.

As an admissible contour may be taken to lie as close to the lines $\operatorname{Im} \varphi(z) = 0$ as desired, the only such limit points z_0 in the whole set

$$\{z \in \mathbb{C} : \operatorname{Re} \varphi(z) > 0\} \cup \{z \in \mathbb{C} : \operatorname{Re} \varphi(z) \le 0 \text{ and } |z| \le 1\}$$

must lie on the curve $\operatorname{Re} \varphi(z_0) = 0$.

Suppose now that z is a zero of $p_{n-1}(r_n z)$ which lies in a sector $|\arg z| \leq \theta - \epsilon$ with $\epsilon > 0$ and in the set

$$\{z \in \mathbb{C} : \operatorname{Re} \varphi(z) \leq 0 \text{ and } |z| > 1\} \setminus N_\epsilon$$

for n large enough. Then z eventually lies outside of γ, and so $F_n(z) = 0$ by (14). From equation (29) it follows that

$$0 = \frac{1}{(1-z)\sqrt{2\pi\lambda n}} + \frac{A\zeta^{1-n} r_n^{b-a}}{(\zeta - z)\sqrt{2\pi\lambda n}} + o\Big(n^{-1/2}\Big) + o\Big(r_n^{b-a} n^{-1/2}\Big)$$

as $n \to \infty$.

If $\operatorname{Re} a > \operatorname{Re} b$ then multiplying through by $\sqrt{n}$ yields

$$0 = \frac{1}{(1-z)\sqrt{2\pi\lambda}} + o(1)$$

as $n \to \infty$, and the only way this is possible is if $|z| \to \infty$. If $\operatorname{Re} a < \operatorname{Re} b$ then multiplying through by $r_n^{a-b}\sqrt{n}$ instead yields

$$0 = \frac{A\zeta^{1-n}}{(\zeta - z)\sqrt{2\pi\lambda}} + o(1)$$

as $n \to \infty$, and we again conclude that we must have $|z| \to \infty$. Finally if $\operatorname{Re} a = \operatorname{Re} b$ then multiplying through by $\sqrt{n}$ yields

$$0 = \left(A^{-1}\frac{\zeta - z}{1-z} + \zeta^{1-n} r_n^{b-a}\right) \frac{1}{(\zeta - z)\sqrt{2\pi\lambda}} + o(1)$$

as $n \to \infty$. Again, the only way this holds is if $|z| \to \infty$.

Since $\epsilon > 0$ was arbitrary, it follows that the zeros of $p_{n-1}(r_n z)$ have no limit point in the set

$$\{z \in \mathbb{C} : \operatorname{Re} \varphi(z) \leq 0 \text{ and } |z| > 1\}.$$

This completes the proof.

4.3 Generalization to more directions of maximal growth

It's not difficult to extend the results in this section to functions with $m+1$, $m \geq 0$, directions of maximal exponential growth as defined in Subsection 2.3.

We can derive an analogue to equations (22) and (29), specifically

$$r_n^a F_n(z) = \frac{1}{\sqrt{2\pi\lambda n}} \sum_{k=0}^{m} \left[\frac{A_k \zeta_k^{1-n} r_n^{b_k}}{\zeta_k - z} + o\Big(r_n^{b_k} n^{-1/2}\Big)\right] \tag{37}$$

as $n \to \infty$ uniformly with respect to z as long as z remains in any sector $|\arg(z/\zeta_k)| \leq \theta - \epsilon$ with $\epsilon > 0$, $k = 0, 1, \ldots, m$, and remains bounded away from γ. Following this we would proceed just as before to conclude that the limit points of the zeros of the scaled partial sum $p_{n-1}(r_n z)$ in the sector $|\arg z| < \theta$, $z \neq 0$ all still lie on the curve

$$\left|z^\lambda \exp\Big(1 - z^\lambda\Big)\right| = 1.$$

If $\operatorname{Re} a > \operatorname{Re} b_k$ for all $k = 1, \ldots, n$ then the zeros will again approach these limit points from the exterior of the curve. The main complication that may arise is if $\operatorname{Re} a$ is equal to a number of the quantities $\operatorname{Re} b_k$, and in that case it may be more difficult to determine where the analogue of formula (32),

$$\left|z^\lambda \exp\Big(1 - z^\lambda\Big)\right| = 1 + \left(\operatorname{Re} a - \operatorname{Re} b_k + \frac{\lambda}{2}\right) \frac{\log n}{n} + O\Big(n^{-1}\Big)$$

for some $k \in \{1, \ldots, m\}$, will hold.

5 Scaling limits at the arcs of the limit curve

In this section we aim to study the zeros of the scaled partial sums $p_{n-1}(r_n z)$ which approach the smooth arcs of the limit curve

$$S = \Big\{ z \in \mathbb{C} : \left|z^\lambda \exp\Big(1 - z^\lambda\Big)\right| = 1 \text{ and } |z| \leq 1 \Big\}$$

in the sector $|\arg z| < \theta$. We will determine how quickly these zeros approach the curve, track their movement and the spacing between them, and ultimately calculate a certain limit of the partial sums depending on an argument which follows the zeros in their approach.

The results in Subsections 5.1, 5.2, and 5.3 can be seen as generalizations of Theorem 2 which was originally obtained by Edrei, Saff, and Varga in their monograph [4].

5.1 One direction of maximal exponential growth

Using the definitions in Subsection 2.1, let f be an entire function with a single direction of maximal exponential growth.

We showed in Section 4 that the limit points of the zeros of the scaled partial sums $p_{n-1}(r_n z)$ in the sector $|\arg z| < \theta$, $z \neq 0$, lie on the curve

$$S = \Big\{ z \in \mathbb{C} : \left|z^\lambda \exp\Big(1 - z^\lambda\Big)\right| = 1 \text{ and } |z| \leq 1 \Big\}.$$

Note that if ξ is a point of S then

$$\operatorname{Re}\Big(\xi^\lambda - 1 - \lambda \log \xi\Big) = 0.$$

Theorem 7 *Let ξ be a point of S with $|\arg \xi| < \theta$, $\xi \neq 1$ and define*

$$\tau = \operatorname{Im}\Big(\xi^\lambda - 1 - \lambda \log \xi\Big).$$

Define the sequence τ_n by the conditions

$$\frac{\tau n}{\lambda} \equiv \tau_n \pmod{2\pi}, \qquad -\pi < \tau_n \leq \pi$$

and let

$$z_n(w) = \xi \left(1 + \frac{\log n}{2(1 - \xi^\lambda) n} - \frac{w - i\tau_n}{(1 - \xi^\lambda) n}\right).$$

Then

$$\lim_{n\to\infty} \frac{p_{n-1}(r_n z_n(w))}{f(r_n z_n(w))} = 1 - \frac{e^{-w}}{\xi^a(1-\xi)\sqrt{2\pi\lambda}}$$

uniformly for w restricted to any compact subset of $\mathbb{C}$.

Remark 3 Theorem 7 gives us precise asymptotics for individual zeros of the scaled partial sums $p_{n-1}(r_n z)$ near a given point ξ on the arcs of the curve S. Details of this are given in Subsection 5.4, where we use that information to verify part (a) of the Modified Saff-Varga Width Conjecture for this class of functions.

5.1.1 Proof of Theorem 7

Let γ_θ be the portion of γ in the sector $|\arg z| \le \theta$ and for $\epsilon > 0$ define N_ϵ to be the set of all points within a distance of ϵ of γ_θ. It was shown in the proof of Theorem 5 (see equation (22)) that

$$F_n(z) = \frac{1}{(1-z)\sqrt{2\pi\lambda n}} + o\Big(n^{-1/2}\Big) \tag{38}$$

as $n \to \infty$ uniformly for $z \in \mathbb{C} \setminus N_\epsilon$ with $|\arg z| \le \theta - \epsilon$ for any fixed $\epsilon > 0$.

Because $\operatorname{Re}\varphi(\xi) = 0$ and because γ is an admissible contour for the function φ, $\epsilon > 0$ can be taken small enough so that

$$\inf_{s\in\gamma} |\xi - s| > \epsilon.$$

Consequently if w is restricted to a compact subset of $\mathbb{C}$ then $z_n(w) \notin N_\epsilon$ and $|\arg z_n(w)| \le \theta - \epsilon$ for all such w if n is large enough. It follows from (38) that

$$\begin{aligned} F_n(z_n(w)) &= \frac{1}{(1-z_n(w))\sqrt{2\pi\lambda n}} + o\Big(n^{-1/2}\Big) \\ &\sim \frac{1}{(1-\xi)\sqrt{2\pi\lambda n}} \end{aligned}$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\mathbb{C}$. Then, since $z_n(w)$ is inside γ for n large enough, (8) implies that

$$\frac{f(r_n z_n(w))}{r_n^a(\log r_n)^b(e^{1/\lambda}z_n(w))^n}\left(\frac{p_{n-1}(r_n z_n(w))}{f(r_n z_n(w))} - 1\right) \sim -\frac{1}{(1-\xi)\sqrt{2\pi\lambda n}} \tag{39}$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\mathbb{C}$.

It follows from the asymptotic assumption on f in (1) that

$$\begin{aligned} \frac{f(r_n z_n(w))}{r_n^a(\log r_n)^b} &\sim \xi^a e^{r_n^\lambda z_n(w)^\lambda} \\ &= \xi^a \exp\left\{\frac{n\xi^\lambda}{\lambda}\left(1 + \frac{\log n}{2(1-\xi^\lambda)n} - \frac{w - i\tau_n}{(1-\xi^\lambda)n}\right)^\lambda\right\} \\ &\sim \xi^a \exp\left\{\frac{n\xi^\lambda}{\lambda}\left(1 + \frac{\lambda\log n}{2(1-\xi^\lambda)n} - \frac{\lambda(w - i\tau_n)}{(1-\xi^\lambda)n}\right)\right\} \\ &= \xi^a n^{\xi^\lambda/[2(1-\xi^\lambda)]} \exp\left\{\frac{n\xi^\lambda}{\lambda} - \frac{\xi^\lambda(w - i\tau_n)}{1-\xi^\lambda}\right\} \end{aligned}$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\mathbb{C}$. Since

$$\begin{aligned}
z_n(w)^n &= \xi^n \left(1 + \frac{\log n}{2(1-\xi^\lambda)n} - \frac{w - i\tau_n}{(1-\xi^\lambda)n}\right)^n \\
&= \xi^n \exp\left\{n \log\left(1 + \frac{\log n}{2(1-\xi^\lambda)n} - \frac{w - i\tau_n}{(1-\xi^\lambda)n}\right)\right\} \\
&\sim \xi^n \exp\left\{\frac{\log n}{2(1-\xi^\lambda)} - \frac{w - i\tau_n}{1-\xi^\lambda}\right\} \\
&= \xi^n n^{1/[2(1-\xi^\lambda)]} \exp\left\{-\frac{w - i\tau_n}{1-\xi^\lambda}\right\}
\end{aligned}$$

it then follows that

$$\begin{aligned}
\frac{f(r_n z_n(w))}{r_n^a (\log r_n)^b (e^{1/\lambda} z_n(w))^n} &\sim \xi^a n^{-1/2} \exp\left\{\frac{n(\xi^\lambda - 1 - \lambda \log \xi)}{\lambda} + w - i\tau_n\right\} \\
&= \xi^a n^{-1/2} \exp\left\{\frac{i\tau n}{\lambda} + w - i\tau_n\right\} \\
&= \xi^a n^{-1/2} e^w \qquad (40)
\end{aligned}$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\mathbb{C}$.

Substituting (40) into (39) yields the limit

$$\frac{p_{n-1}(r_n z_n(w))}{f(r_n z_n(w))} \longrightarrow 1 - \frac{e^{-w}}{\xi^a (1-\xi)\sqrt{2\pi\lambda}}$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\mathbb{C}$, which is exactly the limit we desire.

5.2 Maximal growth in two directions and its effect on the scaling limit

Using the definitions in Subsection 2.2, let f be an entire function with two directions of maximal exponential growth.

We showed in Section 4 that the limit points of the zeros of the scaled partial sums $p_{n-1}(r_n z)$ in the sector $|\arg z| < \theta$, $z \neq 0$, lie on the curve

$$S = \left\{ z \in \mathbb{C} : \left| z^\lambda \exp\left(1 - z^\lambda\right) \right| = 1 \text{ and } |z| \leq 1 \right\}.$$

Note that if ξ is a point of S then

$$\mathrm{Re}\left(\xi^\lambda - 1 - \lambda \log \xi\right) = 0.$$

Theorem 8 *Let ξ be a point of S with $|\arg \xi| < \theta$, $\xi \neq 1$ and define*

$$\tau = \mathrm{Im}\left(\xi^\lambda - 1 - \lambda \log \xi\right).$$

Define the sequence τ_n by the conditions

$$\frac{\tau n}{\lambda} \equiv \tau_n \pmod{2\pi}, \qquad -\pi < \tau_n \leq \pi$$

and let

$$z_n^1(w) = \xi\left(1 + \frac{\log n}{2(1-\xi^\lambda)n} - \frac{w - i\tau_n}{(1-\xi^\lambda)n}\right).$$

Define the sequence σ_n *by the conditions*

$$n \arg\zeta \equiv \sigma_n \pmod{2\pi}, \qquad -\pi < \sigma_n \le \pi$$

and let

$$z_n^2(w) = \xi\left[1 + \left(a - b + \frac{\lambda}{2}\right)\frac{\log n}{\lambda(1-\xi^\lambda)n} - \frac{w - i\sigma_n - i\tau_n}{(1-\xi^\lambda)n}\right].$$

If $\operatorname{Re} a > \operatorname{Re} b$ *then*

$$\lim_{n\to\infty} \frac{p_{n-1}(r_n z_n^1(w))}{f(r_n z_n^1(w))} = 1 - \frac{e^{-w}}{\xi^a(1-\xi)\sqrt{2\pi\lambda}},$$

if $\operatorname{Re} a < \operatorname{Re} b$ *then*

$$\lim_{n\to\infty} \frac{p_{n-1}(r_n z_n^2(w))}{f(r_n z_n^2(w))} = 1 - \frac{A\zeta\lambda^{(a-b)/\lambda}e^{-w}}{\xi^a(\zeta-\xi)\sqrt{2\pi\lambda}},$$

and if $\operatorname{Re} a = \operatorname{Re} b$ *then*

$$\frac{p_{n-1}(r_n z_n^1(w))}{f(r_n z_n^1(w))} = 1 - \left(\frac{1}{1-\xi} + \frac{A\zeta^{1-n}r_n^{b-a}}{\zeta - \xi}\right)\frac{e^{-w}}{\xi^a\sqrt{2\pi\lambda}} + o(1)$$

as $n \to \infty$. *All three limits are uniform with respect to* w *as long as* w *is restricted to a compact subset of* $\mathbb{C}$.

Remark 4 Depending on the balance between $\operatorname{Re} a$ and $\operatorname{Re} b$ there are three possible forms of the scaling limit in this case, compared to only one when f has a single direction of maximal growth as in Subsection 5.1. Examples of these extra scaling limits are given in Section 7 (in all subsections except 7.3).

Remark 5 Just as with Theorem 7 in the case of one direction of maximal growth, Theorem 8 gives us precise asymptotics for individual zeros of the scaled partial sums $p_{n-1}(r_n z)$ near a given point ξ on the arcs of the curve S in the case of two directions of maximal growth. Details of this are given in Subsection 5.4, where we use that information to verify part (a) of the Modified Saff-Varga Width Conjecture for this class of functions.

5.2.1 Proof of Theorem 8

Let γ_θ be the portion of γ in the sector $|\arg z| \le \theta$ and for $\epsilon > 0$ define N_ϵ to be the set of all points within a distance of ϵ of γ_θ. It was shown in the proof of Theorem 6 (see equation (29)) that

$$F_n(z) = \frac{1}{(1-z)\sqrt{2\pi\lambda n}} + \frac{A\zeta^{1-n}r_n^{b-a}}{(\zeta - z)\sqrt{2\pi\lambda n}} + o\left(n^{-1/2}\right) + o\left(r_n^{b-a}n^{-1/2}\right) \tag{41}$$

as $n \to \infty$ uniformly for $z \in \mathbb{C} \setminus N_\epsilon$ with $|\arg z| \le \theta - \epsilon$ for any fixed $\epsilon > 0$.

Fix $j \in \{1,2\}$. Because $\operatorname{Re}\varphi(\xi) = 0$ and because γ is an admissible contour for the function φ, $\epsilon > 0$ can be taken small enough so that

$$\inf_{s\in\gamma} |\xi - s| > \epsilon.$$

Consequently if w is restricted to a compact subset of $\mathbb{C}$ then $z_n^j(w) \notin N_\epsilon$ and $|\arg z_n^j(w)| \le \theta - \epsilon$ for all such w if n is large enough. It follows from (41) that

$$\begin{aligned} F_n(z_n^j(w)) &= \frac{1}{(1 - z_n^j(w))\sqrt{2\pi\lambda n}} + \frac{A\zeta^{1-n} r_n^{b-a}}{(\zeta - z_n^j(w))\sqrt{2\pi\lambda n}} + o\Big(n^{-1/2}\Big) + o\Big(r_n^{b-a} n^{-1/2}\Big) \\ &= \frac{1}{(1-\xi)\sqrt{2\pi\lambda n}} + \frac{A\zeta^{1-n} r_n^{b-a}}{(\zeta - \xi)\sqrt{2\pi\lambda n}} + o\Big(n^{-1/2}\Big) + o\Big(r_n^{b-a} n^{-1/2}\Big) \end{aligned}$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\mathbb{C}$. Then, since $z_n^j(w)$ is inside γ for n large enough, (14) implies that

$$\begin{aligned} &\frac{f(r_n z_n^j(w))}{r_n^a (e^{1/\lambda} z_n^j(w))^n} \left(\frac{p_{n-1}(r_n z_n^j(w))}{f(r_n z_n^j(w))} - 1 \right) \\ &\qquad = -\frac{1}{(1-\xi)\sqrt{2\pi\lambda n}} - \frac{A\zeta^{1-n} r_n^{b-a}}{(\zeta - \xi)\sqrt{2\pi\lambda n}} + o\Big(n^{-1/2}\Big) + o\Big(r_n^{b-a} n^{-1/2}\Big) \end{aligned} \tag{42}$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\mathbb{C}$.

Suppose that $\operatorname{Re} a > \operatorname{Re} b$. Then from (42) it follows that

$$\frac{f(r_n z_n^1(w))}{r_n^a (e^{1/\lambda} z_n^1(w))^n} \left(\frac{p_{n-1}(r_n z_n^1(w))}{f(r_n z_n^1(w))} - 1 \right) \sim -\frac{1}{(1-\xi)\sqrt{2\pi\lambda n}}$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\mathbb{C}$. This is analogous to equation (39) from the proof of Theorem 7, and by proceeding as in that proof it can be shown that

$$\frac{p_{n-1}(r_n z_n^1(w))}{f(r_n z_n^1(w))} \longrightarrow 1 - \frac{e^{-w}}{\xi^a (1-\xi)\sqrt{2\pi\lambda}} \tag{43}$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\mathbb{C}$. This is the first desired limit in Theorem 8.

Now suppose $\operatorname{Re} a < \operatorname{Re} b$. From (42) it follows that

$$\frac{f(r_n z_n^2(w))}{r_n^a (e^{1/\lambda} z_n^2(w))^n} \left(\frac{p_{n-1}(r_n z_n^2(w))}{f(r_n z_n^2(w))} - 1 \right) \sim -\frac{A\zeta^{1-n} r_n^{b-a}}{(\zeta - \xi)\sqrt{2\pi\lambda n}} \tag{44}$$

as $n \to \infty$ uniformly for w restricted to a compact subset of $\mathbb{C}$.

The asymptotic assumption on f in (11) implies that

$$\begin{aligned} \frac{f(r_n z_n^2(w))}{r_n^a} &\sim \xi^a e^{r_n^\lambda z_n^2(w)^\lambda} \\ &= \xi^a \exp\left\{ \frac{n\xi^\lambda}{\lambda} \left[1 + \left(a - b + \frac{\lambda}{2} \right) \frac{\log n}{\lambda(1-\xi^\lambda)n} - \frac{w - i\sigma_n - i\tau_n}{(1-\xi^\lambda)n} \right]^\lambda \right\} \\ &\sim \xi^a \exp\left\{ \frac{n\xi^\lambda}{\lambda} \left[1 + \left(a - b + \frac{\lambda}{2} \right) \frac{\log n}{(1-\xi^\lambda)n} - \frac{\lambda(w - i\sigma_n - i\tau_n)}{(1-\xi^\lambda)n} \right] \right\} \\ &= \xi^a \exp\left\{ \frac{n\xi^\lambda}{\lambda} + \left(a - b + \frac{\lambda}{2} \right) \frac{\xi^\lambda \log n}{\lambda(1-\xi^\lambda)} - \frac{\xi^\lambda (w - i\sigma_n - i\tau_n)}{1-\xi^\lambda} \right\} \end{aligned}$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\mathbb{C}$. Since

$$\begin{aligned}
z_n^2(w)^n &= \xi^n \left[1 + \left(a - b + \frac{\lambda}{2}\right) \frac{\log n}{\lambda(1-\xi^\lambda)n} - \frac{w - i\sigma_n - i\tau_n}{(1-\xi^\lambda)n}\right]^n \\
&= \xi^n \exp\left\{n \log\left[1 + \left(a - b + \frac{\lambda}{2}\right) \frac{\log n}{\lambda(1-\xi^\lambda)n} - \frac{w - i\sigma_n - i\tau_n}{(1-\xi^\lambda)n}\right]\right\} \\
&\sim \xi^n \exp\left\{n \left[\left(a - b + \frac{\lambda}{2}\right) \frac{\log n}{\lambda(1-\xi^\lambda)n} - \frac{w - i\sigma_n - i\tau_n}{(1-\xi^\lambda)n}\right]\right\} \\
&= \xi^n \exp\left\{\left(a - b + \frac{\lambda}{2}\right) \frac{\log n}{\lambda(1-\xi^\lambda)} - \frac{w - i\sigma_n - i\tau_n}{1-\xi^\lambda}\right\}
\end{aligned}$$

it then follows that

$$\begin{aligned}
\frac{f(r_n z_n^2(w))}{r_n^a (e^{1/\lambda} z_n^2(w))^n} &\sim \xi^a n^{(b-a)/\lambda - 1/2} \exp\left\{\frac{n}{\lambda}\left(\xi^\lambda - 1 - \lambda \log \xi\right) + w - i\sigma_n - i\tau_n\right\} \\
&= \xi^a n^{(b-a)/\lambda - 1/2} \exp\left\{\frac{i\tau n}{\lambda} + w - i\sigma_n - i\tau_n\right\} \\
&= \xi^a n^{(b-a)/\lambda - 1/2} e^{w - i\sigma_n} \qquad (45)
\end{aligned}$$

as $n \to \infty$ uniformly for w restricted to a compact subset of $\mathbb{C}$.

Substituting (45) into (44) yields the limit

$$\frac{p_{n-1}(r_n z_n^2(w))}{f(r_n z_n^2(w))} \longrightarrow 1 - \frac{A\zeta^{1-n}\lambda^{(a-b)/\lambda} e^{i\sigma_n - w}}{\xi^a(\zeta - \xi)\sqrt{2\pi\lambda}} = 1 - \frac{A\zeta\lambda^{(a-b)/\lambda} e^{-w}}{\xi^a(\zeta - \xi)\sqrt{2\pi\lambda}}$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\mathbb{C}$. This is the second desired limit in Theorem 8.

Finally suppose $\operatorname{Re} a = \operatorname{Re} b$. In this case, after setting $j = 1$ equation (42) becomes

$$\begin{aligned}
&\frac{f(r_n z_n^1(w))}{r_n^a (e^{1/\lambda} z_n^1(w))^n} \left(\frac{p_{n-1}(r_n z_n^1(w))}{f(r_n z_n^1(w))} - 1\right) \\
&\qquad = -\left(\frac{1}{1-\xi} + \frac{A\zeta^{1-n} r_n^{b-a}}{\zeta - \xi}\right) \frac{1}{\xi^a \sqrt{2\pi\lambda n}} + o\left(n^{-1/2}\right)
\end{aligned}$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\mathbb{C}$. By following the same method as in the proof of Theorem 7 to obtain equation (40) it can be shown that

$$\frac{f(r_n z_n^1(w))}{r_n^a (e^{1/\lambda} z_n^1(w))^n} \sim \xi^a n^{-1/2} e^w$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\mathbb{C}$. Substituting this into the above yields the asymptotic

$$\frac{p_{n-1}(r_n z_n^1(w))}{f(r_n z_n^1(w))} = 1 - \left(\frac{1}{1-\xi} + \frac{A\zeta^{1-n} r_n^{b-a}}{\zeta - \xi}\right) \frac{e^{-w}}{\xi^a \sqrt{2\pi\lambda}} + o(1)$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\mathbb{C}$, which is the last desired item in Theorem 8.

5.3 Generalization to more directions of maximal growth

Let f be an entire function with $m+1$ directions of maximal exponential growth as defined in Subsection 2.3.

As in Subsection 4.3 we can derive an analogue to equations (22) and (29), specifically

$$F_n(z) = \frac{1}{\sqrt{2\pi\lambda n}} \sum_{k=0}^{m} \left[\frac{A_k \zeta_k^{1-n} r_n^{b_k - a}}{\zeta_k - z} + o\Big(r_n^{b_k - a} n^{-1/2} \Big) \right] \tag{46}$$

as $n \to \infty$ uniformly with respect to z as long as z remains in any sector $|\arg(z/\zeta_k)| \leq \theta - \epsilon$ with $\epsilon > 0$, $k = 0, 1, \ldots, m$, and remains bounded away from γ.

The main difficulty in extending Theorem 8 to arbitrary $m \geq 2$ lies in the necessity of modifying the quantity $z_n(w)$ that appears in the scaling limit when the real parts of the b_k balance in different ways. In the simplest case there is a j such that $\operatorname{Re} b_j > \operatorname{Re} b_k$ for all $k \neq j$, which allows us to simplify (37) into

$$F_n(z) \sim \frac{A_j \zeta_j^{1-n} r_n^{b_j - a}}{(\zeta_j - z)\sqrt{2\pi\lambda n}},$$

after which the analysis proceeds just as in the relevant parts of the proof of Theorem 8. This yields the following result.

Theorem 9 *Let $m \geq 1$ and suppose that there is a $j \in \{0, 1, \ldots, m\}$ such that $\operatorname{Re} b_j > \operatorname{Re} b_k$ for all $k \neq j$. Let ξ be a point of*

$$S = \Big\{ z \in \mathbb{C} : \Big| z^\lambda \exp\Big(1 - z^\lambda\Big) \Big| = 1 \text{ and } |z| \leq 1 \Big\}$$

with $|\arg \xi| < \theta$, $\xi \neq 1$ and define

$$\tau = \operatorname{Im}\Big(\xi^\lambda - 1 - \lambda \log \xi \Big).$$

Define the sequences τ_n and σ_n by the conditions

$$\frac{\tau n}{\lambda} \equiv \tau_n \pmod{2\pi}, \qquad -\pi < \tau_n \leq \pi,$$

$$n \arg \zeta_j \equiv \sigma_n \pmod{2\pi}, \qquad -\pi < \sigma_n \leq \pi$$

and let

$$z_n(w) = \xi \left[1 + \left(a - b_j + \frac{\lambda}{2} \right) \frac{\log n}{\lambda(1 - \xi^\lambda) n} - \frac{w - i\sigma_n - i\tau_n}{(1 - \xi^\lambda) n} \right].$$

Then

$$\lim_{n \to \infty} \frac{p_{n-1}(r_n z_n(w))}{f(r_n z_n(w))} = 1 - \frac{A_j \zeta_j \lambda^{(a - b_j)/\lambda} e^{-w}}{\xi^a (\zeta_j - \xi) \sqrt{2\pi\lambda}}$$

uniformly for w restricted to any compact subset of $\mathbb{C}$.

Now we will consider what happens when $\operatorname{Re} a$ is not strictly larger than all of the other $\operatorname{Re} b_k$. Because the analysis for general m would be very complicated, we will restrict ourselves to the case $m = 2$. Up to relabeling the b_k there are three cases not included in the theorem above: (i) $\operatorname{Re} a = \operatorname{Re} b_1 > \operatorname{Re} b_2$, (ii) $\operatorname{Re} a = \operatorname{Re} b_1 = \operatorname{Re} b_2$, and (iii) $\operatorname{Re} b_1 = \operatorname{Re} b_2 > \operatorname{Re} a$.

Case (i): $\operatorname{Re} a = \operatorname{Re} b_1 > \operatorname{Re} b_2$**.** In this case (37) becomes

$$F_n(z) = \frac{1}{(1-z)\sqrt{2\pi\lambda n}} + \frac{A_1 \zeta_1^{1-n} r_n^{b_1-a}}{(\zeta_1 - z)\sqrt{2\pi\lambda n}} + o\Big(n^{-1/2}\Big).$$

If we define

$$z_n(w) = \xi\left(1 + \frac{\log n}{2(1-\xi^\lambda)n} - \frac{w - i\tau_n}{(1-\xi^\lambda)n}\right)$$

then

$$F_n(z_n(w)) = \left(\frac{1}{1-\xi} + \frac{A_1 \zeta_1^{1-n} r_n^{b_1-a}}{\zeta_1 - \xi}\right)\frac{1}{\sqrt{2\pi\lambda n}} + o\Big(n^{-1/2}\Big)$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\mathbb{C}$. The remainder of the analysis proceeds just as in the analogous part of the proof of Theorem 8 and produces the following result.

Theorem 10 *Let $m = 2$ and $\operatorname{Re} a = \operatorname{Re} b_1 > \operatorname{Re} b_2$. Let ξ be a point of S with $|\arg\xi| < \theta$, $\xi \neq 1$ and define τ and τ_n as in Theorem 9. Let*

$$z_n(w) = \xi\left(1 + \frac{\log n}{2(1-\xi^\lambda)n} - \frac{w - i\tau_n}{(1-\xi^\lambda)n}\right).$$

Then

$$\frac{p_{n-1}(r_n z_n(w))}{f(r_n z_n(w))} = 1 - \left(\frac{1}{1-\xi} + \frac{A_1 \zeta_1^{1-n} r_n^{b_1-a}}{\zeta_1 - \xi}\right)\frac{e^{-w}}{\xi^a\sqrt{2\pi\lambda}} + o(1)$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\mathbb{C}$.

Case (ii): $\operatorname{Re} a = \operatorname{Re} b_1 = \operatorname{Re} b_2$**.** This case is very similar to the previous one, the only difference being that we keep all three terms in the sum in (37) instead of only two. Doing so yields the following result.

Theorem 11 *Let $m = 2$ and $\operatorname{Re} a = \operatorname{Re} b_1 = \operatorname{Re} b_2$. Let ξ be a point of S with $|\arg\xi| < \theta$, $\xi \neq 1$ and define τ and τ_n as in Theorem 9 and $z_n(w)$ as in Theorem 10. Then*

$$\frac{p_{n-1}(r_n z_n(w))}{f(r_n z_n(w))} = 1 - \left(\frac{1}{1-\xi} + \frac{A_1 \zeta_1^{1-n} r_n^{b_1-a}}{\zeta_1 - \xi} + \frac{A_2 \zeta_2^{1-n} r_n^{b_2-a}}{\zeta_2 - \xi}\right)\frac{e^{-w}}{\xi^a\sqrt{2\pi\lambda}} + o(1)$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\mathbb{C}$.

Case (iii): $\operatorname{Re} b_1 = \operatorname{Re} b_2 > \operatorname{Re} a$**.** In this case we retain both the b_1 term and the b_2 term in (37), i.e.

$$F_n(z) = \left(\frac{A_1}{\zeta_1 - z} + \frac{A_2(\zeta_2/\zeta_1)^{1-n} r_n^{b_2-b_1}}{\zeta_2 - z}\right)\frac{\zeta_1^{1-n} r_n^{b_1-a}}{\sqrt{2\pi\lambda n}} + o\Big(r_n^{b_1-a} n^{-1/2}\Big).$$

Unfortunately we can't easily cancel the oscillations coming from the factor in parentheses by adding extra periodicity to $z_n(w)$ like before. We will again have to settle for an asymptotic rather than a proper limit.

Theorem 12 *Let* $m = 2$ *and* $\operatorname{Re} b_1 = \operatorname{Re} b_2 > \operatorname{Re} a$. *Let* ξ *be a point of* S *with* $|\arg \xi| < \theta$, $\xi \neq 1$ *and define* τ *and* τ_n *as in Theorem 9. Define the sequence* σ_n *by the conditions*

$$n \arg \zeta_1 \equiv \sigma_n \pmod{2\pi}, \qquad -\pi < \sigma_n \leq \pi$$

and let

$$z_n(w) = \xi \left[1 + \left(a - b_1 + \frac{\lambda}{2}\right) \frac{\log n}{\lambda(1-\xi^\lambda)n} - \frac{w - i\sigma_n - i\tau_n}{(1-\xi^\lambda)n}\right].$$

Then

$$\frac{p_{n-1}(r_n z_n(w))}{f(r_n z_n(w))} = 1 - \left(\frac{A_1}{\zeta_1 - \xi} + \frac{A_2(\zeta_2/\zeta_1)^{1-n} r_n^{b_2-b_1}}{\zeta_2 - \xi}\right) \frac{\zeta_1 \lambda^{(a-b_1)/\lambda} e^{-w}}{\xi^a \sqrt{2\pi\lambda}} + o(1)$$

as $n \to \infty$ *uniformly for* w *restricted to any compact subset of* $\mathbb{C}$.

5.4 Verification of the Modified Saff-Varga Width Conjecture in the sector $0 < |\arg z| < \theta$

The theorems in this section allow us to verify part (a) of the Modified Saff-Varga Width Conjecture for the function f in the sector $0 < |\arg z| < \theta$.

When f has a single direction of maximal exponential growth, Theorem 7 and Hurwitz's theorem imply that if ξ is any point of the curve

$$S = \left\{z \in \mathbb{C} : \left|z^\lambda \exp\left(1 - z^\lambda\right)\right| = 1 \text{ and } |z| \leq 1\right\}$$

with $0 < |\arg \xi| < \theta$ and if w_0 is any solution to the equation

$$\xi^a (1-\xi)\sqrt{2\pi\lambda} = e^{-w} \tag{47}$$

then $p_{n-1}(z)$ has a zero z_0 satisfying

$$z_0 = r_n \xi \left(1 + \frac{\log n}{2(1-\xi^\lambda)n} - \frac{w_0 - i\tau_n}{(1-\xi^\lambda)n} + o\left(n^{-1}\right)\right) \tag{48}$$

as $n \to \infty$.

Fix $\phi \in (-\theta, \theta)$ with $\phi \neq 0$. There is a unique $\xi \in S$ such that $\xi = |\xi| e^{i\phi}$. Set $\rho_n = |\xi| r_n$. Then for the zero z_0 above,

$$\begin{aligned} z_0 - \rho_n e^{i\phi} &= z_0 - r_n \xi \\ &\sim r_n \xi \frac{\log n}{2(1-\xi^\lambda)n} \\ &= \rho_n e^{i\phi} \frac{\log n}{2(1-\xi^\lambda)n} \end{aligned}$$

as $n \to \infty$. It follows that, for any $\epsilon > 0$, z_0 will lie inside the disk

$$\left|z - \rho_n e^{i\phi}\right| \leq \rho_n n^{-1+\epsilon}$$

for n large enough. As equation (47) has infinitely many solutions, the number of zeros of $p_{n-1}(z)$ in any such disk will tend to infinity as $n \to \infty$. Since $\rho_n =$

$\Theta(n^{1/\lambda}) = O(n^{2/\lambda})$ this completes the verification of part (a) of the Modified Saff-Varga Width Conjecture for the sector $0 < |\arg z| < \theta$.

When f has two directions of maximal exponential growth and $\operatorname{Re} a \neq \operatorname{Re} b$ then the situation is very similar to the one above. Theorem 8 implies that for any $\xi \in S$ with $0 < |\arg \xi| < \theta$ there is an equation of the form

$$u = e^w$$

for some $u = u(\xi) \in \mathbb{C}$ such that if w_0 is any solution of the equation then $p_{n-1}(z)$ has a zero z_0 of the form

$$z_0 = r_n \xi + C r_n \xi \frac{\log n}{n} + h_n(w_0),$$

where $C \in \mathbb{C}$ and h_n is a function satisfying $h_n(z) = O(n^{-1})$ as $n \to \infty$ for any fixed $z \in \mathbb{C}$. Part (b) of the Modified Saff-Varga Width Conjecture follows just as above.

When f has two directions of maximal exponential growth and $\operatorname{Re} a = \operatorname{Re} b$ then Theorem 8 implies that for any $\xi \in S$ with $0 < |\arg \xi| < \theta$ it is possible to find a constant $D \in \mathbb{C}$ with $D \neq 0$ and a subsequence M such that

$$\lim_{m \in M} \frac{p_{m-1}(r_m z_m^1(w))}{f(r_m z_m^1(w))} = 1 - De^{-w}$$

uniformly on compact subsets of the w-plane. So if w_0 is any solution of the equation

$$1 = De^{-w}$$

then by Hurwitz's theorem $p_{m-1}(z)$ has a zero z_0 of the form

$$z_0 = r_m \xi \left(1 + \frac{\log m}{2(1-\xi^\lambda)m} - \frac{w_0 - i\tau_m}{(1-\xi^\lambda)m} + o\Big(m^{-1}\Big) \right)$$

as $m \to \infty$ with $m \in M$. The rest of the verification of the Modified Saff-Varga Width Conjecture proceeds just as above, though with the indices restricted to the subsequence M (as allowed in the Conjecture).

By using Theorems 9, 10, 11, and 12 we can verify the conjecture for functions with three directions of maximal exponential growth, and though we haven't obtained any explicit results in the case that f has more than three directions of maximal exponential growth the conjecture can be verified using a similar method.

6 Scaling limits at the corner of the limit curve

In this section we aim to study the zeros of the scaled partial sums $p_n(r_n z)$ which approach the corner of the limit curve

$$S = \left\{ z \in \mathbb{C} : \left| z^\lambda \exp\Big(1 - z^\lambda\Big) \right| = 1 \text{ and } |z| \leq 1 \right\}$$

located at $z = 1$. To this end we will calculate a certain limit of the partial sums depending on an argument which follows the zeros as they approach this corner.

The results in Subsections 6.1, 6.2, and 6.3 can be seen as generalizations of Theorem 1 which was obtained by Edrei, Saff, and Varga in their monograph [4].

6.1 One direction of maximal exponential growth

Using the definitions in Subsection 2.1, let f be an entire function with a single direction of maximal exponential growth.

Theorem 13

$$\lim_{n\to\infty} \frac{p_{n-1}(r_n(1+w/\sqrt{n}))}{f(r_n(1+w/\sqrt{n}))} = \frac{1}{2}\operatorname{erfc}\Big(w\sqrt{\lambda/2}\Big)$$

uniformly for w restricted to any compact subset of $\operatorname{Re} w < 0$.

The function erfc in the theorem statement above is known as the complementary error function and is defined by

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-s^2}\, ds, \tag{49}$$

where the contour of integration is the horizontal line starting at $s = z$ and extending to the right to $s = z + \infty$. Information about the zeros of this function can be found in [5].

Remark 6 Theorem 13 gives us precise asymptotics for individual zeros of the scaled partial sums $p_{n-1}(r_n z)$ near the corner of the limit curve S located at $z = 1$. Details of this are given in Subsection 6.4, where we use that information to verify part (b) of the Modified Saff-Varga Width Conjecture for this class of functions.

6.1.1 Proof of Theorem 13

Choose $\epsilon > 0$ such that $\overline{B_{2\epsilon}(1)} \subset U$ and define

$$m(z) = \begin{cases} G_n(z) & \text{for } z \in \mathbb{C} \setminus \Big(\gamma_\theta \cup \overline{B_{2\epsilon}(1)}\Big), \\ G_n(z) - P_n(z) & \text{for } z \in B_{2\epsilon}(1) \setminus \gamma_\theta. \end{cases}$$

The jumps for $G_n(z)$ and $P_n(z)$ cancel each other out as z moves across γ_θ in $B_{2\epsilon}(1)$, so m is analytic on $B_{2\epsilon}(1)$. If we define the contours

$$\Gamma_1 = \partial B_{2\epsilon}(1), \qquad \Gamma_2 = \gamma_\theta \setminus B_{2\epsilon}(1), \qquad \Gamma = \Gamma_1 \cup \Gamma_2, \tag{50}$$

where Γ_1 is oriented in the counterclockwise direction and Γ_2 inherits its orientation from γ_θ (and thus γ), then the function m uniquely solves the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 2. *Seek an analytic function $M\colon \mathbb{C} \setminus \Gamma \to \mathbb{C}$ such that*

1. $M^+(z) = M^-(z) - P_n(z)$ *for $z \in \Gamma_1 \setminus \Gamma_2$,*
2. $M^+(z) = M^-(z) + e^{n\varphi(z)}$ *for $z \in \Gamma_2$ except at endpoints,*
3. $M(z) \to 0$ *as $|z| \to \infty$,*
4. *if c is an endpoint of either arc of Γ_2 then $M(z) = O(|z-c|^q)$ as $z \to c$ with $z \in \mathbb{C} \setminus \Gamma_2$ for some $q > -1$.*

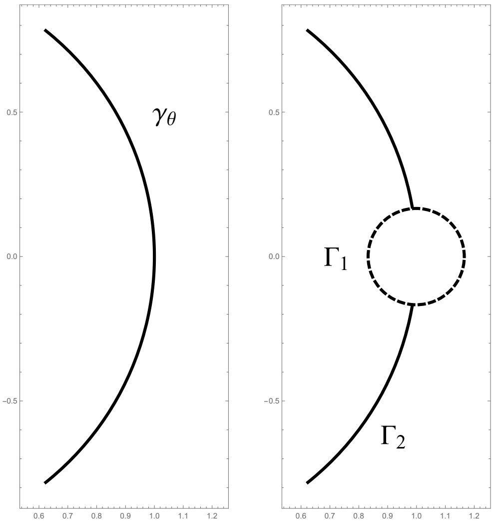

Fig. 3 Schematic representations of the curve γ_θ (left) and the curve Γ in (50) (right). In the plot of Γ, the curve Γ_1 is dashed and the curve Γ_2 is solid black.

Plemelj's formula then yields

$$\begin{aligned} m(z) &= \frac{1}{2\pi i} \int_\Gamma \Big[m^+(s) - m^-(s) \Big] \frac{ds}{s-z} \\ &= -\frac{1}{2\pi i} \int_{\Gamma_1} P_n(s) \frac{ds}{s-z} + \frac{1}{2\pi i} \int_{\Gamma_2} e^{n\varphi(s)} \frac{ds}{s-z}. \end{aligned} \tag{51}$$

Lemma 5

$$\int_{\Gamma_1} P_n(s) \frac{ds}{s-z} = O(n^{-1/2})$$

uniformly for $z \in B_\epsilon(1)$ as $n \to \infty$.

Proof. There exists a constant C_1 such that

$$|h(\zeta)| \leq C_1 |\zeta|^{-1}$$

for $\zeta \notin \mathbb{R}$. Setting $\zeta = -i\sqrt{n}\psi^{-1}(s)$ yields

$$|P_n(s)| \leq C_1 n^{-1/2} |\psi^{-1}(s)|^{-1} = C_1 n^{-1/2} |\varphi(s)|^{-1/2}$$

for $s \in U \setminus \gamma_\theta$. Thus if $s \in \Gamma_1$ then $|\varphi(s)| \geq C_2$ for some constant $C_2 > 0$, so

$$\left| \int_{\Gamma_1} P_n(s) \frac{ds}{s-z} \right| \leq C_1 C_2^{-1/2} \epsilon^{-1} n^{-1/2}$$

for $z \in B_\epsilon(1)$. □

Lemma 6 *There is a constant $c > 0$ such that*

$$\int_{\Gamma_2} e^{n\varphi(s)} \frac{ds}{s-z} = O(e^{-cn})$$

uniformly for $z \in B_\epsilon(1)$ as $n \to \infty$.

Proof. Recalling Definition 1, since $\gamma_\theta = \gamma \cap \{z \in \mathbb{C}\colon |\arg z| \le \theta\}$ and $\Gamma_2 = \gamma_\theta \setminus B_{2\epsilon}(1)$ there exists a constant $c > 0$ such that $\operatorname{Re}\varphi(s) < -c$ for $s \in \Gamma_2$. Further, if $z \in B_\epsilon(1)$ and $s \in \Gamma_2$ then $|s - z| > \epsilon$, so that

$$\begin{aligned}\left|\int_{\Gamma_2} e^{n\varphi(s)} \frac{ds}{s-z}\right| &\le \int_{\Gamma_2} e^{n \operatorname{Re}\varphi(s)} \frac{|ds|}{|s-z|} \\ &< \epsilon^{-1} \operatorname{length}(\Gamma_2) e^{-cn}.\end{aligned}$$

□

Combining Lemmas 5 and 6 yields

$$m(z) = o(1),$$

and hence, by the definition of m,

$$G_n(z) = P_n(z) + o(1) \tag{52}$$

uniformly for $z \in B_\epsilon(1)$ as $n \to \infty$. Now set $z = 1 + w/\sqrt{n}$, where w is restricted to a compact subset of $\operatorname{Re} w < 0$.

Lemma 7

$$\lim_{n\to\infty} F_n\big(1 + w/\sqrt{n}\big) - G_n\big(1 + w/\sqrt{n}\big) = 0$$

uniformly for w restricted to compact subsets of $\operatorname{Re} w < 0$.

Proof. In this proof we will write $z = 1 + w/\sqrt{n}$ as a shorthand, keeping in mind the implicit dependence of z on w and n.

Split the integral for F_n into the two pieces

$$F_n(z) = \frac{r_n^{-a}(\log r_n)^{-b}}{2\pi i}\left(\int_{\gamma_\theta} + \int_{\gamma\setminus\gamma_\theta}\right)(e^{1/\lambda}s)^{-n} f(r_n s)\frac{ds}{s-z}. \tag{53}$$

As in the previous lemma the second integral here is uniformly exponentially decreasing.

By the asymptotic assumption on f in (1), for $|\arg z| \le \theta$ we can write

$$f(z) = z^a (\log z)^b \exp(z^\lambda)\big[1 + \delta(z)\big], \tag{54}$$

where $\delta(z) \to 0$ uniformly as $|z| \to \infty$ in this sector. This implies

$$\frac{f(r_n s)}{r_n^a (\log r_n)^b (e^{1/\lambda}s)^n} = s^a e^{n\varphi(s)}\left(1 + \frac{\log s}{\log r_n}\right)^b \big[1 + \delta(r_n s)\big]$$

for $s \in \gamma_\theta$. Then define

$$\tilde{\delta}(r_n, s) = \left(1 + \frac{\log s}{\log r_n}\right)^b \big[1 + \delta(r_n s)\big] - 1 \tag{55}$$

and write the integrand of the first integral in (53) as

$$e^{n\varphi(s)} + e^{n\varphi(s)}\,(s^a - 1) + s^a e^{n\varphi(s)}\tilde{\delta}(r_n, s).$$

Recalling the definition of G_n in (9), it follows that (53) can be rewritten as

$$F_n(z) = G_n(z) + \frac{1}{2\pi i} \int_{\gamma_\theta} e^{n\varphi(s)} (s^a - 1) \frac{ds}{s-z} + \frac{1}{2\pi i} \int_{\gamma_\theta} s^a e^{n\varphi(s)} \tilde{\delta}(r_n, s) \frac{ds}{s-z} + O(e^{-cn}) \tag{56}$$

for some constant $c > 0$. We will show that both of these remaining integrals tend to 0 uniformly.

The contour γ_θ passes through the point $s = 1$ vertically, so by assumption there exists a positive constant C_1 such that $|s - z| \geq C_1 n^{-1/2}$. For n large enough $z \notin \gamma_\theta$, and in that case

$$\left| \frac{s-1}{s-z} \right| \leq 1 + \left| \frac{1-z}{s-z} \right| \leq 1 + C_1^{-1} n^{1/2} |1 - z| \leq C_2$$

for some constant C_2. Hence

$$\begin{aligned} \left| \int_{\gamma_\theta} e^{n\varphi(s)} (s^a - 1) \frac{ds}{s-z} \right| &\leq \int_{\gamma_\theta} e^{n \operatorname{Re} \varphi(s)} \left| \frac{s^a - 1}{s-1} \right| \left| \frac{s-1}{s-z} \right| |ds| \\ &\leq C_2 \int_{\gamma_\theta} e^{n \operatorname{Re} \varphi(s)} \left| \frac{s^a - 1}{s-1} \right| |ds|, \end{aligned}$$

which tends to zero as $n \to \infty$.

Split the second integral in (56) like

$$\int_{\gamma_\theta} = \int_{\gamma_\theta \cap B_\epsilon(1)} + \int_{\gamma_\theta \setminus B_\epsilon(1)}.$$

The integral over $\gamma_\theta \setminus B_\epsilon(1)$ decreases exponentially. Let $s = \psi(it)$ and let

$$-i\psi^{-1}(\gamma_\theta \cap B_\epsilon(1)) = (-\alpha_1, \alpha_2),$$

where $\alpha_1, \alpha_2 > 0$, so that

$$\begin{aligned} &\left| \int_{\gamma_\theta \cap B_\epsilon(1)} s^a e^{n\varphi(s)} \tilde{\delta}(r_n, s) \frac{ds}{s-z} \right| \\ &\quad = \left| \int_{-\alpha_1}^{\alpha_2} e^{-nt^2} \tilde{\delta}(r_n, \psi(it)) \frac{\psi(it)^a \psi'(it)}{\psi(it) - z} \, dt \right| \\ &\quad \leq C_1^{-1} n^{1/2} \sup_{-\alpha_1 < t < \alpha_1} \left| \tilde{\delta}(r_n, \psi(it)) \psi(it)^a \psi'(it) \right| \int_{-\alpha_1}^{\alpha_2} e^{-nt^2} \, dt \\ &\quad < C_1^{-1} \sqrt{\pi} \sup_{-\alpha_1 < t < \alpha_1} \left| \tilde{\delta}(r_n, \psi(it)) \psi(it)^a \psi'(it) \right|, \end{aligned}$$

which tends to 0 as $n \to \infty$ by the assumption on δ and, by extension, $\tilde{\delta}$.

Combining the above estimates with (56) it follows that

$$F_n(z) = G_n(z) + o(1)$$

uniformly as $n \to \infty$. □

As a consequence of this lemma, equation (52) implies that

$$F_n\big(1+w/\sqrt{n}\big) = P_n\big(1+w/\sqrt{n}\big) + o(1) \tag{57}$$

uniformly as $n \to \infty$.

Following the argument in [10, p. 194], it can be shown that

$$h(\zeta) = \frac{1}{2}e^{-\zeta^2}\operatorname{erfc}(-i\zeta) \tag{58}$$

on $\operatorname{Im}\zeta > 0$. Setting

$$\zeta = -i\sqrt{n}\psi^{-1}(z) = -i\sqrt{n\varphi(z)}$$

for an appropriately chosen branch of the square root yields an expression for P_n,

$$P_n(z) = \frac{1}{2}e^{n\varphi(z)}\operatorname{erfc}\Big(-\sqrt{n\varphi(z)}\Big),$$

valid for $z \in U$ to the left of γ_θ. Since $2-\operatorname{erfc}(x) = \operatorname{erfc}(-x)$ this can be rewritten as

$$P_n(z) = e^{n\varphi(z)} - \frac{1}{2}e^{n\varphi(z)}\operatorname{erfc}\Big(\sqrt{n\varphi(z)}\Big).$$

It is straightforward to show that

$$\lim_{n\to\infty} n\varphi(1+w/\sqrt{n}) = \frac{\lambda}{2}w^2$$

uniformly, so

$$P_n(1+w/\sqrt{n}) = e^{\lambda w^2/2} - \frac{1}{2}e^{\lambda w^2/2}\operatorname{erfc}\Big(w\sqrt{\lambda/2}\Big) + o(1)$$

uniformly as $n \to \infty$. By substituting this into (57) it follows that

$$F_n(1+w/\sqrt{n}) = e^{\lambda w^2/2} - \frac{1}{2}e^{\lambda w^2/2}\operatorname{erfc}\Big(w\sqrt{\lambda/2}\Big) + o(1) \tag{59}$$

uniformly as $n \to \infty$.

For n large enough

$$F_n(1+w/\sqrt{n}) = \frac{1}{r_n^a(\log r_n)^b}\left(\frac{f(r_n(1+w/\sqrt{n}))}{e^{n/\lambda}(1+w/\sqrt{n})^n} - \frac{p_{n-1}(r_n(1+w/\sqrt{n}))}{e^{n/\lambda}(1+w/\sqrt{n})^n}\right)$$

by (8). The asymptotic assumption (1) grants us the uniform estimate

$$\frac{f(r_n(1+w/\sqrt{n}))}{r_n^a(\log r_n)^b e^{n/\lambda}(1+w/\sqrt{n})^n} = e^{\lambda w^2/2} + o(1),$$

and substituting this into the above formula yields

$$F_n(1+w/\sqrt{n}) = e^{\lambda w^2/2} - e^{\lambda w^2/2}\frac{p_{n-1}(r_n(1+w/\sqrt{n}))}{f(r_n(1+w/\sqrt{n}))}(1+o(1)) + o(1)$$

uniformly as $n \to \infty$. Substituting this into (59) then yields the expression

$$\frac{p_{n-1}(r_n(1+w/\sqrt{n}))}{f(r_n(1+w/\sqrt{n}))}(1+o(1)) = \frac{1}{2}\operatorname{erfc}\Big(w\sqrt{\lambda/2}\Big) + o(1),$$

which holds uniformly as $n \to \infty$. Theorem 13 follows immediately from this asymptotic.

6.2 Maximal growth in two directions and its effect on the scaling limit

In Subsection 5.2 the scaling limit at the arcs of the limit curve for the partial sums of a function with two directions of maximal growth had to be modified depending on the balance of the constants $\operatorname{Re} a$ and $\operatorname{Re} b$. When $\operatorname{Re} a > \operatorname{Re} b$ one limit holds, when $\operatorname{Re} a < \operatorname{Re} b$ another, and when $\operatorname{Re} a = \operatorname{Re} b$ a sort of transitional limit holds.

The behavior of the scaling limit at the corner of the limit curve is starkly different. It does not go through any such change when $\operatorname{Re} a = \operatorname{Re} b$. In a sense the geometry of the zeros of the partial sums doesn't change that much when $\operatorname{Re} a - \operatorname{Re} b$ is only slightly negative compared to when it is slightly positive—the zeros still lie outside the curve—and it turns out that the corner scaling limit isn't sensitive to the change that does happen there, which is that the rate at which the zeros approach the arcs of the limit curve begins to depend on both a and b (see Theorem 8, compare the definitions of $z_n^1(w)$ and $z_n^2(w)$).

There is a second bifurcation past this one which we have noticed previously in Section 4 (Theorem 6), namely that the zeros approach the limit curve from the exterior when $\operatorname{Re} a - \operatorname{Re} b > -\lambda/2$ and from the interior when $\operatorname{Re} a - \operatorname{Re} b < -\lambda/2$. We will see that this bifurcation is so severe that the scaling limit at the corner of the limit curve completely changes from one case to the other. Indeed, this corner scaling limit in the case $\operatorname{Re} a - \operatorname{Re} b > -\lambda/2$ will again involve the complementary error function just as in Subsection 6.1, but once $\operatorname{Re} a - \operatorname{Re} b < -\lambda/2$ the scaling limit will take the form $1 + De^{-w}$. Further, the scaling we choose in this latter case will approach the corner of the curve more slowly by a factor of $\sqrt{\log n}$.

For examples of this behavior see Subsections 7.4 and 7.6.

6.2.1 Statement of the result

Using the definitions in Subsection 2.2, let f be an entire function with two directions of maximal exponential growth.

Theorem 14 *If* $\operatorname{Re} b - \operatorname{Re} a < \lambda/2$ *then*

$$\lim_{n\to\infty} \frac{p_{n-1}(r_n(1+w/\sqrt{n}))}{f(r_n(1+w/\sqrt{n}))} = \frac{1}{2}\operatorname{erfc}\Big(w\sqrt{\lambda/2}\Big) \tag{60}$$

uniformly for w restricted to any compact subset of $\operatorname{Re} w < 0$.

If $\operatorname{Re} b - \operatorname{Re} a = \lambda/2$ *then*

$$\frac{p_{n-1}(r_n(1+w/\sqrt{n}))}{f(r_n(1+w/\sqrt{n}))} = \frac{1}{2}\operatorname{erfc}\Big(w\sqrt{\lambda/2}\Big) - \frac{A\zeta^{1-n} r_n^{b-a} e^{-\lambda w^2/2}}{(\zeta-1)\sqrt{2\pi\lambda n}} + o(1) \tag{61}$$

as $n \to \infty$ uniformly for w restricted to any compact subset of $\operatorname{Re} w < 0$.

Define the sequence σ_n by the conditions

$$n \arg\zeta \equiv \sigma_n \pmod{2\pi}, \qquad -\pi < \sigma_n \le \pi$$

and let

$$z_n(w) = 1 - \sqrt{\frac{\kappa \log n}{n}} - \frac{w - i\sigma_n}{\lambda\sqrt{\kappa n \log n}}, \tag{62}$$

where

$$\kappa := \frac{2}{\lambda}\left(\frac{b-a}{\lambda} - \frac{1}{2}\right)$$

and the principal branch of the square root is taken.

If $\operatorname{Re} b - \operatorname{Re} a > \lambda/2$ *then*

$$\lim_{n\to\infty} \frac{p_{n-1}(r_n z_n(w))}{f(r_n z_n(w))} = 1 - \frac{A\zeta\lambda^{(a-b)/\lambda}e^{-w}}{(\zeta-1)\sqrt{2\pi\lambda}} \tag{63}$$

uniformly as long as w is restricted to a compact subset of $\mathbb{C}$.

Remark 7 Just as with Theorem 13 in the case of one direction of maximal growth, Theorem 14 gives us precise asymptotics for individual zeros of the scaled partial sums $p_{n-1}(r_n z)$ near the corner of the limit curve S located at $z = 1$. Details of this are given in Subsection 6.4, where we use that information to verify part (b) of the Modified Saff-Varga Width Conjecture for this class of functions.

6.2.2 Proof of Theorem 14

As in the proof of Theorem 6 in Subsection 4.2.1 we will split γ into three parts. Call γ_1 the part of γ in the sector $|\arg z| \leq \theta$, call γ_2 the part of γ in the sector $|\arg(z/\zeta)| \leq \theta$, and call γ_3 the part of γ outside of either of those sectors. Then divide the integral in (13) into the three parts

$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}.$$

Using a method identical to the proof of Lemma 1 it can be shown that

$$\int_{\gamma_3} \left(e^{1/\lambda}s\right)^{-n} f(r_n s)\frac{ds}{s-z} = O\left(e^{(\mu-1)n/\lambda}\right) \tag{64}$$

as $n \to \infty$ uniformly for z restricted to any sector $|\arg z| \leq \theta - \epsilon$ with $\epsilon > 0$. Also, Lemma 4 grants the asymptotic

$$\int_{\gamma_2} \left(e^{1/\lambda}s\right)^{-n} f(r_n s)\frac{ds}{s-z} = \frac{iA\zeta^{1-n}r_n^b}{\zeta - z}\sqrt{\frac{2\pi}{\lambda n}} + o\left(r_n^b n^{-1/2}\right) \tag{65}$$

as $n \to \infty$ uniformly for $z \in \mathbb{C} \setminus \tilde{N}_\epsilon$ with $\epsilon > 0$.

Combining equations (64) and (65) in the definition of F_n yields the estimate

$$F_n(z) = \frac{r_n^{-a}}{2\pi i}\int_{\gamma_1} \left(e^{1/\lambda}s\right)^{-n} f(r_n s)\frac{ds}{s-z} + \frac{A\zeta^{1-n}r_n^{b-a}}{(\zeta-z)\sqrt{2\pi\lambda n}} + o\left(r_n^b n^{-1/2}\right) \tag{66}$$

as $n \to \infty$ uniformly for z restricted to any sector $|\arg z| \leq \theta - \epsilon$ with $\epsilon > 0$.

Case 1: $\operatorname{Re} b - \operatorname{Re} a \leq \lambda/2$. Using essentially the same technique used in Subsection 6.1.1 to prove Theorem 13 it can be shown that

$$\frac{r_n^{-a}}{2\pi i}\int_{\gamma_1} \left(e^{1/\lambda}s\right)^{-n} f(r_n s)\frac{ds}{s-z}\bigg|_{z=1+w/\sqrt{n}} = e^{\lambda w^2/2} - \frac{1}{2}e^{\lambda w^2/2}\operatorname{erfc}\left(w\sqrt{\lambda/2}\right) + o(1)$$

and that

$$F_n(1+w/\sqrt{n}) = e^{\lambda w^2/2} - e^{\lambda w^2/2}\frac{p_{n-1}(r_n(1+w/\sqrt{n}))}{f(r_n(1+w/\sqrt{n}))}\big[1+o(1)\big] + o(1)$$

as $n \to \infty$ uniformly for w restricted to compact subsets of $\operatorname{Re} w < 0$. Substituting these into (66) produces the estimate

$$\begin{aligned}\frac{p_{n-1}(r_n(1+w/\sqrt{n}))}{f(r_n(1+w/\sqrt{n}))}&\big[1+o(1)\big]\\ &= \frac{1}{2}\operatorname{erfc}\Big(w\sqrt{\lambda/2}\Big) - \frac{A\zeta^{1-n}r_n^{b-a}e^{-\lambda w^2/2}}{(\zeta-1-w/\sqrt{n})\sqrt{2\pi\lambda n}} + o\Big(r_n^b n^{-1/2}\Big) + o(1)\end{aligned}$$

as $n \to \infty$ uniformly for w restricted to compact subsets of $\operatorname{Re} w < 0$. Since $r_n = (n/\lambda)^{1/\lambda}$, assuming $\operatorname{Re} b - \operatorname{Re} a < \lambda/2$ yields (60) and assuming $\operatorname{Re} b - \operatorname{Re} a = \lambda/2$ yields (61).

Case 2: $\operatorname{Re} b - \operatorname{Re} a > \lambda/2$. In this case we will use the scaling $z = z_n(w)$ from (62) instead of the scaling $z = 1 + w/\sqrt{n}$ from the previous case. The integral over γ_1 needs a new estimate.

Lemma 8

$$\frac{r_n^{-a}}{2\pi i}\int_{\gamma_1}\Big(e^{1/\lambda}s\Big)^{-n} f(r_n s)\frac{ds}{s-z}\bigg|_{z=z_n(w)} = o\Big(r_n^{b-a}n^{-1/2}\Big)$$

uniformly as $n \to \infty$ with w restricted to any compact subset of $\mathbb{C}$.

Proof. By the asymptotic assumption on f in (11), for $|\arg z| \le \theta$ we can write

$$f(z) = z^a \exp(z^\lambda)\big[1+\delta(z)\big], \tag{67}$$

where $\delta(z) \to 0$ uniformly as $|z| \to \infty$ in this sector. Hence

$$\frac{f(r_n s)}{r_n^a (e^{1/\lambda}s)^n} = s^a e^{n\varphi(s)}\big[1+\delta(r_n s)\big],$$

and $\delta(r_n s) = o(1)$ uniformly as $n \to \infty$ for $s \in \gamma_1$. We can then write

$$\frac{r_n^{-a}}{2\pi i}\int_{\gamma_1}\Big(e^{1/\lambda}s\Big)^{-n} f(r_n s)\frac{ds}{s-z} = G_n(z) + \frac{1}{2\pi i}\int_{\gamma_1} s^a e^{n\varphi(s)}\delta(r_n s)\frac{ds}{s-z},$$

where $G_n(z)$ is as defined in equation (9). The argument at the end of Lemma 7 shows that

$$\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\gamma_1} s^a e^{n\varphi(s)}\delta(r_n s)\frac{ds}{s-z}\bigg|_{z=z_n(w)} = 0,$$

uniformly, and equation (52) ensures that $G_n(z) = P_n(z) + o(1)$ for z sufficiently close to $z = 1$, where P_n is as defined in equation (10), so it just remains to estimate $P_n(z_n(w))$.

Let $q_n(w) = -i\sqrt{n}\psi^{-1}(z_n(w))$, where ψ is as defined in equation (5). We calculate

$$\psi^{-1}(x) = \sqrt{\lambda/2}(x-1) + O\Big((x-1)^2\Big)$$

as $x \to 1$, so

$$-q_n(w)^2 = \left(\frac{b-a}{\lambda} - \frac{1}{2}\right)\log n + O(1)$$

and

$$-iq_n(w) \sim \sqrt{\frac{\lambda\kappa}{2}\log n}$$

uniformly as $n \to \infty$. By assumption $\operatorname{Re}\kappa > 0$, thus $|\arg\sqrt{\kappa}| < \pi/4$. It follows that $\operatorname{erfc}(-iq_n(w)) \to 0$ and hence, combining equations (10) and (58),

$$P_n(z) = \frac{1}{2}e^{-q_n(w)^2}\operatorname{erfc}(-iq_n(w)) = o\Big(n^{(b-a)/\lambda-1/2}\Big) = o\Big(r_n^{b-a}n^{-1/2}\Big)$$

uniformly as $n \to \infty$. □

In light of this lemma, in this case equation (66) implies that

$$F_n(z_n(w)) = \frac{A\zeta^{1-n}r_n^{b-a}}{(\zeta - z_n(w))\sqrt{2\pi\lambda n}} + o\Big(r_n^{b-a}n^{-1/2}\Big) \tag{68}$$

uniformly as $n \to \infty$ with w restricted to any compact subset of $\mathbb{C}$.

For n large enough,

$$F_n(z_n(w)) = \frac{f(r_n z_n(w))}{r_n^a e^{n/\lambda} z_n(w)^n}\left(1 - \frac{p_{n-1}(r_n z_n(w))}{f(r_n z_n(w))}\right) \tag{69}$$

by (14). Following a similar calculation to those leading to (40) and (45), the asymptotic assumption (11) grants us the uniform estimate

$$\frac{f(r_n z_n(w))}{r_n^a e^{n/\lambda} z_n(w)^n} \sim n^{(b-a)/\lambda-1/2}e^{w-i\sigma_n},$$

and combining this with (68) and (69), noting that $\zeta - z_n(w) \to \zeta - 1$, yields

$$\frac{p_{n-1}(r_n z_n(w))}{f(r_n z_n(w))} = 1 - \frac{A\zeta\lambda^{(a-b)/\lambda}e^{-w}}{(\zeta-1)\sqrt{2\pi\lambda}} + o(1)$$

uniformly as $n \to \infty$ with w restricted to any compact subset of $\mathbb{C}$. This is precisely equation (63).

This completes the proof of Theorem 14.

6.3 Generalization to more directions of maximal growth

Unlike the scaling limit at the arcs of the limit curve (Subsection 5.3), generalizing the scaling limit at the corner of the curve is relatively straightforward.

Using the definitions in Subsection 2.3, let f be an entire function with $m+1$ directions of maximal exponential growth.

For each new direction of maximal growth of f, equation (66) gains a corresponding term and error term. Explicitly, for f as defined above, the equation becomes

$$F_n(z) = \frac{r_n^{-a}}{2\pi i}\int_{\gamma_1}\left(e^{1/\lambda}s\right)^{-n} f(r_n s)\frac{ds}{s-z} + \sum_{k=1}^{m}\left[\frac{A_k\zeta_k^{1-n}r_n^{b_k-a}}{(\zeta_k - z)\sqrt{2\pi\lambda n}} + o(r_n^{b_k-a}n^{-1/2})\right]$$

as $n \to \infty$ uniformly for z restricted to any sector $|\arg z| \le \theta - \epsilon$ with $\epsilon > 0$. Following the argument in the proof of Theorem 14 presents the following result.

Theorem 15 *Let $J \subset \{1,2,\ldots,m\}$ be the set of indices j satisfying $\operatorname{Re} b_j = \max\{\operatorname{Re} b_1, \operatorname{Re} b_2, \ldots, \operatorname{Re} b_m\}$.*

If $\operatorname{Re} b_k - \operatorname{Re} a \le \lambda/2$ for all $k = 1,2,\ldots,m$ then

$$\frac{p_{n-1}(r_n(1+w/\sqrt{n}))}{f(r_n(1+w/\sqrt{n}))} = \frac{1}{2}\operatorname{erfc}\Big(w\sqrt{\lambda/2}\Big) - \frac{e^{-\lambda w^2/2}}{\sqrt{2\pi\lambda n}} \sum_{k\in J} \frac{A_k \zeta_k^{1-n} r_n^{b_k-a}}{\zeta_k - 1} + o(1) \tag{70}$$

as $n \to \infty$ uniformly when w restricted to any compact subset of $\operatorname{Re} w < 0$.

Now suppose there is a $j \in J$ such that $\operatorname{Re} b_j - \operatorname{Re} a > \lambda/2$. Define the sequence σ_n by the conditions

$$n \arg \zeta_j \equiv \sigma_n \pmod{2\pi}, \qquad -\pi < \sigma_n \le \pi$$

and let

$$z_n(w) = 1 - \sqrt{\frac{\kappa \log n}{n}} - \frac{w - i\sigma_n}{\lambda\sqrt{\kappa n \log n}}, \tag{71}$$

where

$$\kappa := \frac{2}{\lambda}\left(\frac{b_j - a}{\lambda} - \frac{1}{2}\right)$$

and the principal branch of the square root is taken. Then

$$\frac{p_{n-1}(r_n z_n(w))}{f(r_n z_n(w))} = 1 - \frac{\zeta_j \lambda^{(a-b_j)/\lambda} e^{-w}}{\sqrt{2\pi\lambda}} \sum_{k\in J} \frac{A_k (\zeta_k/\zeta_j)^{1-n} r_n^{b_k-b_j}}{\zeta_k - 1} + o(1) \tag{72}$$

uniformly as $n \to \infty$ when w is restricted to any compact subset of $\mathbb{C}$.

Remark 8 Theorem 15 gives us precise asymptotics for individual zeros of the scaled partial sums $p_{n-1}(r_n z)$ near the corner of the limit curve S located at $z = 1$ in the case where f has multiple directions of maximal exponential growth. Details of this are given in Subsection 6.4, where we use that information to verify part (b) of the Modified Saff-Varga Width Conjecture for this class of functions.

6.4 Verification of the Modified Saff-Varga Width Conjecture at the exceptional argument $\arg z = 0$

The theorems in this section allow us to verify part (b) of the Modified Saff-Varga Width Conjecture at the exceptional argument $\arg z = 0$.

First suppose f has a single direction of maximal exponential growth or that f has multiple directions of maximal growth and that $\operatorname{Re} b_k - \operatorname{Re} a < \lambda/2$, $k = 1,\ldots,m$. Under these conditions Theorems 13, 14, and 15 imply that if $w = w_0$ is any solution of the equation

$$\operatorname{erfc}\Big(w\sqrt{\lambda/2}\Big) = 0 \tag{73}$$

then $p_{n-1}(z)$ has a zero $z = z_0$ of the form

$$z_0 = r_n + r_n w_0/\sqrt{n} + o\Big(r_n n^{-1/2}\Big)$$

as $n \to \infty$. It follows that

$$z_0 - r_n \sim r_n w_0/\sqrt{n}$$

as $n \to \infty$, and hence that, for any fixed $\epsilon > 0$, z_0 lies in the disk

$$|z - r_n| \le r_n n^{-1/2+\epsilon}$$

for n large enough. Since equation (73) has infinitely-many solutions, the number of zeros of $p_{n-1}(z)$ in any such disk tends to infinity as $n \to \infty$. Setting $\rho_n = r_n$, this is precisely the condition in part (b) of the Modified Saff-Varga Width Conjecture with $k = 2$ at the exceptional argument $\arg z = 0$.

Now suppose f has two or more directions of maximal exponential growth and let $J \subset \{1, 2, \ldots, m\}$ be the set of indices j satisfying

$$\operatorname{Re} b_j = \max\{\operatorname{Re} b_1, \operatorname{Re} b_2, \ldots, \operatorname{Re} b_m\}.$$

If $\operatorname{Re} b_j - \operatorname{Re} a = \lambda/2$ for $j \in J$, then Theorems 14 and 15 imply that there is a subsequence M and a constant D such that

$$\lim_{\substack{n\to\infty \\ n\in M}} \frac{p_{n-1}(r_n(1 + w/\sqrt{n}))}{f(r_n(1 + w/\sqrt{n}))} = \frac{1}{2}\operatorname{erfc}\Big(w\sqrt{\lambda/2}\Big) - De^{-\lambda w^2/2}$$

uniformly on compact subsets of $\operatorname{Re} w < 0$. So if $w = w_0$ is any solution of the equation

$$\frac{1}{2}\operatorname{erfc}\Big(w\sqrt{\lambda/2}\Big) - De^{-\lambda w^2/2} = 0$$

then, by Hurwitz's theorem, $p_{n-1}(z)$ has a zero $z = z_0$ of the form

$$z_0 = r_n + r_n w_0/\sqrt{n} + o\Big(r_n n^{-1/2}\Big)$$

as $n \to \infty$ with $n \in M$. The rest of the verification follows just as above, though with the indices restricted to the subsequence M (as allowed in the Conjecture).

Finally suppose there is a $j \in J$ such that $\operatorname{Re} b_j - \operatorname{Re} a > \lambda/2$. Define the sequence σ_n by the conditions

$$n \arg \zeta_j \equiv \sigma_n \pmod{2\pi}, \qquad -\pi < \sigma_n \le \pi$$

and let

$$z_n(w) = 1 - \sqrt{\frac{\kappa \log n}{n}} - \frac{w - i\sigma_n}{\lambda\sqrt{\kappa n \log n}},$$

where

$$\kappa := \frac{2}{\lambda}\left(\frac{b_j - a}{\lambda} - \frac{1}{2}\right)$$

and the principal branch of the square root is taken. Theorems 14 and 15 imply that there is a subsequence M and a constant D such that

$$\lim_{\substack{n\to\infty \\ n\in M}} \frac{p_{n-1}(r_n z_n(w))}{f(r_n z_n(w))} = 1 - De^{-w}$$

uniformly on compact subsets of the w-plane. If $w = w_0$ is any solution of the equation

$$1 = De^{-w} \tag{74}$$

then, by Hurwitz's theorem, $p_{n-1}(z)$ has a zero $z = z_0$ of the form

$$z_0 = r_n z_n(w_0 + o(1)) = r_n \left[1 - \sqrt{\frac{\kappa \log n}{n}} - \frac{w_0 - i\sigma_n}{\lambda\sqrt{\kappa n \log n}} + o\Big((n \log n)^{-1/2}\Big) \right]$$

as $n \to \infty$ with $n \in M$. It follows that

$$z_0 - r_n \sim -r_n \sqrt{\frac{\kappa \log n}{n}}$$

as $n \to \infty$ with $n \in M$, and hence that, for any $\epsilon > 0$, z_0 lies inside the disk

$$|z - r_n| \leq r_n n^{-1/2+\epsilon}$$

for $n \in M$ large enough. As equation (74) has infinitely-many solutions, the number of zeros of $p_{n-1}(z)$ in any such disk tends to infinity as $n \to \infty$. Setting $\rho_n = r_n$, this is again precisely the condition in part (b) of the Modified Saff-Varga Width Conjecture with $k = 2$ at the exceptional argument $\arg z = 0$.

7 Applications

In this section we will apply the results from Sections 4, 5, and 6 to several common special functions. These functions were chosen to illustrate different behaviors of the zeros of the partial sums.

7.1 Sine and cosine

Each of the functions

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \text{and} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

are entire of order 1 and have two directions of maximal exponential growth, one as $|z| \to \infty$ with $\epsilon < \arg z < \pi - \epsilon$ and one as $|z| \to \infty$ with $-\pi + \epsilon < \arg z < -\epsilon$. Indeed, if $\theta \in (0, \pi/2)$ then

$$\left.\begin{aligned} &\left| \frac{\mp 2i \sin(\pm iz)}{e^z} - 1 \right| \\ &\left| \frac{2\cos(\pm iz)}{e^z} - 1 \right| \end{aligned}\right\} \leq e^{-2|z|\cos\theta}, \qquad |\arg z| \leq \theta$$

and

$$\left.\begin{aligned} &|\mp 2i \sin(\pm iz)| \\ &|2\cos(\pm iz)| \end{aligned}\right\} \leq 2e^{|z|\cos\theta}, \qquad \theta \leq |\arg z| \leq \pi - \theta.$$

So, for $\theta \in (0, \pi/2)$ and $\mu = \cos\theta$ we have

$$\mp 2i \sin(\pm iz) = \begin{cases} e^z\,[1 + o(1)] & \text{for } |\arg z| \leq \theta, \\ -e^{-z}\,[1 + o(1)] & \text{for } |\arg -z| \leq \theta, \\ O\Big(e^{\mu|z|}\Big) & \text{otherwise} \end{cases}$$

and

$$2\cos(\pm iz) = \begin{cases} e^{z}\,[1+o(1)] & \text{for } |\arg z| \le \theta, \\ e^{-z}\,[1+o(1)] & \text{for } |\arg -z| \le \theta, \\ O\left(e^{\mu|z|}\right) & \text{otherwise} \end{cases}$$

as $|z| \to \infty$ uniformly in each of these sectors.

In the notation of the asymptotic assumption on the functions f we have considered in this paper (see equation (11)), for each of the rotated and scaled functions $\mp 2i\sin(\pm iz)$ and $2\cos(\pm iz)$ and for $\theta \in (0, \pi/2)$ we have $a = b = 0$, $\lambda = 1$, $\mu = \cos\theta$, and $\zeta = -1$. For the sine functions we have $A = -1$ and for the cosines we have $A = 1$.

Remark 9 Applying the results in this paper to the functions $-2i\sin(-iz)$ and $2\cos(-iz)$ gives us information about the zeros of the partial sums of $\sin z$ and $\cos z$ in the upper half-plane, and applying them to the function $2i\sin(iz)$ and $2\cos(-iz)$ gives information about the lower half-plane. This is a very useful trick; by applying the results repeatedly to rotated versions of a function with multiple directions of maximal growth we can obtain information about its partial sums in all of its maximal growth sectors. This technique is used throughout this section.

Applying Theorems 6, 8, and 14 to these sine and cosine functions yields the following collections of results.

Theorem 16 *Let*

$$p_n[\sin](z) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

denote the n^{th} partial sum of the Maclaurin series for $\sin z$ *and let*

$$S = \{z \in \mathbb{C} : |z\exp(1-z)| = 1,\ |z| \le 1,\ \textit{and}\ \operatorname{Re} z > 0\}.$$

The limit points of the zeros of the scaled partial sums $p_{n-1}[\sin](nz)$ which do not lie on the real axis are precisely the points of the set $iS \cup -iS$.

Let $\xi \in S$, $\xi \ne 1$ and define

$$\tau = \operatorname{Im}(\xi - 1 - \log\xi),$$

$$\tau_n \equiv \tau n \pmod{2\pi}, \quad -\pi < \tau_n \le \pi,$$

and

$$z_n(w) = \xi\left(1 + \frac{\log n}{2(1-\xi)n} - \frac{w - i\tau_n}{(1-\xi)n}\right).$$

Then

$$\frac{p_{n-1}[\sin](\pm inz_n(w))}{\sin(\pm inz_n(w))} = 1 - \left(\frac{1}{1-\xi} - \frac{(-1)^n}{1+\xi}\right)\frac{e^{-w}}{\sqrt{2\pi}} + o(1) \tag{75}$$

as $n \to \infty$ uniformly on compact subsets of the w-plane.

Additionally,

$$\lim_{n\to\infty} \frac{p_{n-1}[\sin](\pm i(n + w\sqrt{n}))}{\sin(\pm i(n + w\sqrt{n}))} = \frac{1}{2}\operatorname{erfc}\left(\frac{w}{\sqrt{2}}\right) \tag{76}$$

uniformly on compact subsets of $\operatorname{Re} w < 0$.

Theorem 17 *Let*

$$p_n[\cos](z) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k z^{2k}}{(2k)!}$$

denote the n^{th} *partial sum of the Maclaurin series for* $\cos z$ *and let*

$$S = \{z \in \mathbb{C} : |z \exp(1-z)| = 1,\ |z| \leq 1,\ \textit{and}\ \operatorname{Re} z > 0\}.$$

The limit points of the zeros of the scaled partial sums $p_{n-1}[\cos](nz)$ *which do not lie on the real axis are precisely the points of the set* $iS \cup -iS$*.*

Let $\xi \in S$*,* $\xi \neq 1$ *and define*

$$\tau = \operatorname{Im}(\xi - 1 - \log \xi),$$

$$\tau_n \equiv \tau n \pmod{2\pi}, \quad -\pi < \tau_n \leq \pi,$$

and

$$z_n(w) = \xi \left(1 + \frac{\log n}{2(1-\xi)n} - \frac{w - i\tau_n}{(1-\xi)n} \right).$$

Then

$$\frac{p_{n-1}[\cos](\pm i n z_n(w))}{\cos(\pm i n z_n(w))} = 1 - \left(\frac{1}{1-\xi} + \frac{(-1)^n}{1+\xi} \right) \frac{e^{-w}}{\sqrt{2\pi}} + o(1)$$

as $n \to \infty$ *uniformly on compact subsets of the* w*-plane.*

Additionally,

$$\lim_{n\to\infty} \frac{p_{n-1}[\cos](\pm i(n + w\sqrt{n}))}{\cos(\pm i(n + w\sqrt{n}))} = \frac{1}{2} \operatorname{erfc}\left(\frac{w}{\sqrt{2}} \right)$$

uniformly on compact subsets of $\operatorname{Re} w < 0$*.*

Remark 10 Due to the appearance of $(-1)^n$ in the scaling limits corresponding to the arcs of the limit curve we actually get two different limits if we restrict n to run through only even or only odd integers. For example, the scaling limit for the sine function yields

$$\lim_{n\to\infty} \frac{p_{2n-1}[\sin](\pm i 2n z_{2n}(w))}{\sin(\pm i 2n z_{2n}(w))} = 1 - \left(\frac{1}{1-\xi} - \frac{1}{1+\xi} \right) \frac{e^{-w}}{\sqrt{2\pi}}$$

and

$$\lim_{n\to\infty} \frac{p_{2n}[\sin](\pm i(2n+1) z_{2n+1}(w))}{\sin(\pm i(2n+1) z_{2n+1}(w))} = 1 - \left(\frac{1}{1-\xi} + \frac{1}{1+\xi} \right) \frac{e^{-w}}{\sqrt{2\pi}},$$

each converging uniformly on compact subsets of the w-plane.

Remark 11 The approximations in Region B in Figure 4 appear much better than those in Region A. Indeed, since the sine function is so close to an exponential in those regions we expect from what is known about the zeros of the partial sums of the exponential function that the absolute error of the approximations is on the order of $(\log n)^2/n^2$ in Region B and on the order of $1/n$ in Region A. In fact, based on the asymptotic expansion obtained in [10] we expect that the two approximations shown in Region A have absolute errors of approximately 0.02 and 0.04, respectively, which agrees with what is shown in the plot.

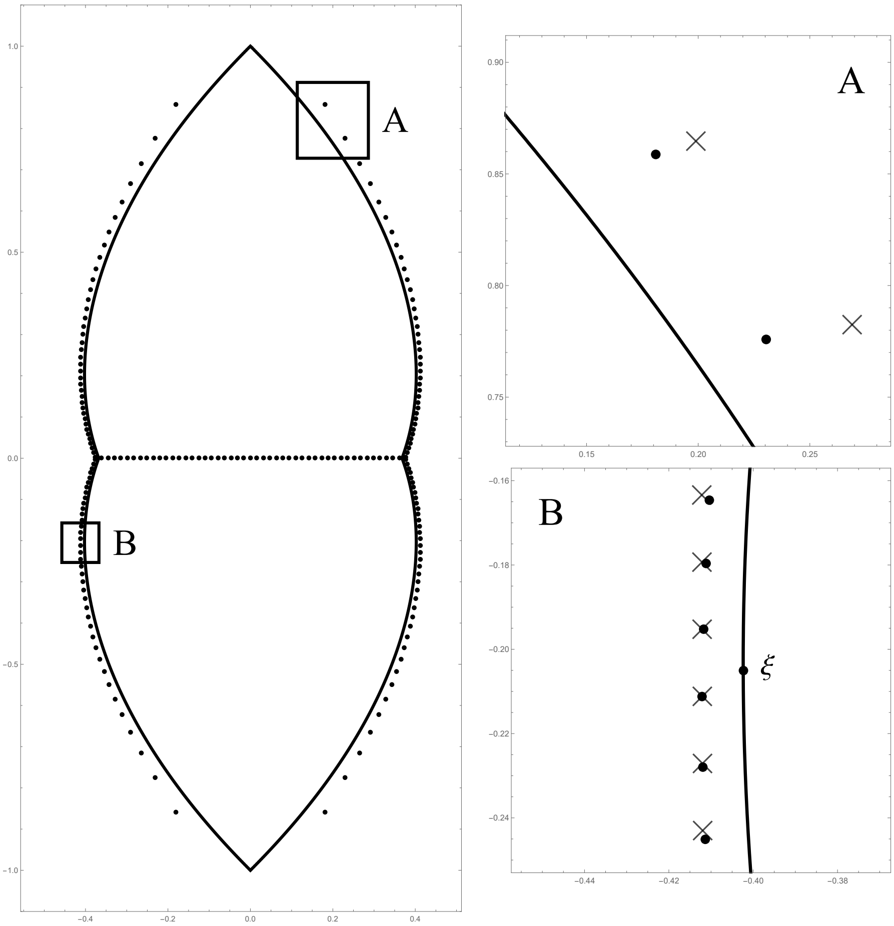

Fig. 4 The zeros of $p_{n-1}[\sin](nz)$ for $n = 200$ and their limit curve. Region A, magnified top-right, shows the approximations for the zeros in that region, represented as crosses, which are given by the corner scaling limit in (76). Region B, magnified bottom-right, shows the approximations in that region given by the curve scaling limit in (75) using the indicated reference point ξ.

7.2 Bessel functions of the first kind

The Bessel functions of the first kind are defined by

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{\Gamma(\nu+k+1)k!}$$

for $\nu \in \mathbb{C}$, where a suitable branch cut is chosen for the factor $(z/2)^\nu$. The series in this definition converges for all $z \in \mathbb{C}$, so the function $(2/z)^\nu J_\nu(z)$ is entire. As such we define the new function

$$\mathbf{J}_\nu(z) = \left(\frac{2}{z}\right)^\nu J_\nu(z).$$

According to NIST's Digital Library of Mathematical Functions [2, eq. 10.17.3] the Bessel function obeys the asymptotic

$$\mathbf{J}_\nu(z) = \frac{1}{\sqrt{\pi}} \left(\frac{2}{z}\right)^{\nu+1/2} \left[\cos\omega\big[1+o(1)\big] + O\!\left(\frac{\sin\omega}{\omega}\right)\right]$$

as $|z| \to \infty$ uniformly in any sector $|\arg z| \le \pi - \epsilon$ with $\epsilon > 0$, where

$$\omega = z - \frac{\pi\nu}{2} - \frac{\pi}{4}.$$

As $\mathbf{J}_\nu(z)$ is even, the same asymptotic is valid for $\mathbf{J}_\nu(-z)$ in $|\arg z| \le \pi - \epsilon$. It follows that for any $\theta \in (0, \pi/2)$ there is a $\mu < 1$ such that

$$\mathbf{J}_\nu(\pm iz) = 2^\nu \sqrt{\frac{2}{\pi}} \times \begin{cases} z^{-\nu-1/2} e^z\,[1+o(1)] & \text{for } |\arg z| \le \theta, \\ (-z)^{-\nu-1/2} e^{-z}\,[1+o(1)] & \text{for } |\arg -z| \le \theta, \\ O\!\left(e^{\mu|z|}\right) & \text{otherwise} \end{cases}$$

as $|z| \to \infty$ uniformly in each of these sectors.

From this information we deduce that the function $\mathbf{J}_\nu$ has two directions of maximal exponential growth. In the notation of the asymptotic assumption on the functions f we have considered in this paper (see equation (11)) we have $a = b = -\nu - 1/2$, $A = 1$ (after rescaling), $\lambda = 1$, and $\zeta = -1$. Applying Theorems 6, 8, and 14 to these functions yields the following collection of results.

Theorem 18 *Let*

$$p_n[\mathbf{J}_\nu](z) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-z^2/4)^k}{\Gamma(\nu+k+1)k!}$$

denote the n^{th} partial sum of the Maclaurin series for $\mathbf{J}_\nu(z)$ and let

$$S = \{z \in \mathbb{C} : |z\exp(1-z)| = 1,\ |z| \le 1,\ \textit{and}\ \operatorname{Re} z > 0\}.$$

The limit points of the zeros of the scaled partial sums $p_{n-1}[\mathbf{J}_\nu](nz)$ which do not lie on the real axis are precisely the points of the set $iS \cup -iS$.

Let $\xi \in S$, $\xi \ne 1$ and define

$$\tau = \operatorname{Im}(\xi - 1 - \log\xi),$$

$$\tau_n \equiv \tau n \pmod{2\pi}, \quad -\pi < \tau_n \le \pi,$$

and

$$z_n(w) = \xi\left(1 + \frac{\log n}{2(1-\xi)n} - \frac{w - i\tau_n}{(1-\xi)n}\right).$$

Then

$$\frac{p_{n-1}[\mathbf{J}_\nu](\pm inz_n(w))}{\mathbf{J}_\nu(\pm inz_n(w))} = 1 - \left(\frac{1}{1-\xi} + \frac{(-1)^n}{1+\xi}\right)\frac{\xi^{\nu+1/2}e^{-w}}{\sqrt{2\pi}} + o(1) \tag{77}$$

as $n \to \infty$ uniformly on compact subsets of the w-plane.

Additionally,

$$\lim_{n\to\infty} \frac{p_{n-1}[\mathbf{J}_\nu](\pm i(n + w\sqrt{n}))}{\mathbf{J}_\nu(\pm i(n + w\sqrt{n}))} = \frac{1}{2}\operatorname{erfc}\!\left(\frac{w}{\sqrt{2}}\right) \tag{78}$$

uniformly on compact subsets of $\operatorname{Re} w < 0$.

Remark 12 Just as in the case of the sine and cosine functions, the scaling limit corresponding to the arcs of the limit curve gives two different limits if we restrict n to run through only even or only odd integers.

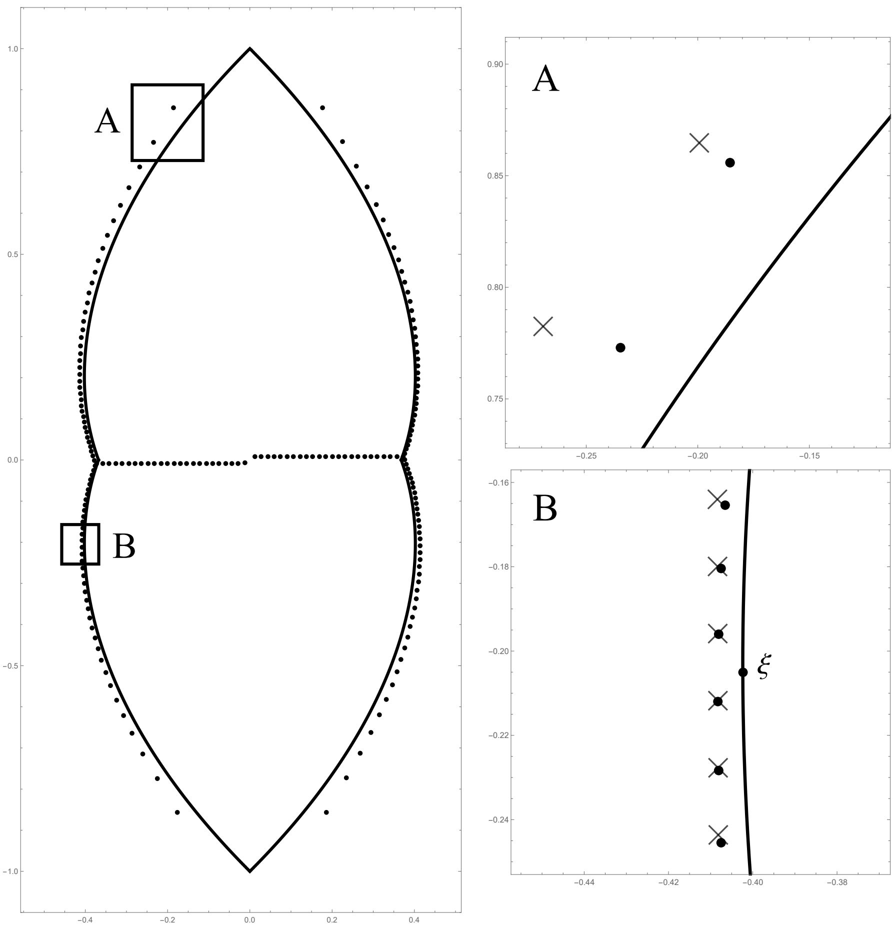

Fig. 5 The zeros of $p_{n-1}[\mathbf{J}_\nu](nz)$ for $\nu = i$ and $n = 200$ and their limit curve. Region A, magnified top-right, shows the approximations for the zeros in that region, represented as crosses, which are given by the corner scaling limit in (78). Region B, magnified bottom-right, shows the approximations in that region given by the curve scaling limit in (77) using the indicated reference point ξ.

7.3 Confluent hypergeometric functions

Here we will consider the functions

$$\mathbf{M}(\alpha, \beta, z) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{\Gamma(k+\beta)k!} z^k,$$

where $\alpha, \beta \in \mathbb{C}$ and $\alpha \neq 0, -1, -2, \ldots$. The series converges for all $z \in \mathbb{C}$, so these functions are entire. They are related to the usual hypergeometric ${}_1F_1$ functions by

$$\frac{{}_1F_1(\alpha,\beta,z)}{\Gamma(\beta)} = \mathbf{M}(\alpha,\beta,z)$$

for fixed $\beta \neq 0, -1, 2, \ldots$, as well as in the limit $\beta \to -m$, $m = 0, 1, 2, \ldots$.

The Digital Library of Mathematical Functions (or the DLMF) gives the following asymptotic for $\mathbf{M}$:

$$\mathbf{M}(\alpha,\beta,z) = \frac{z^{\alpha-\beta}e^z}{\Gamma(\alpha)}\big[1+o(1)\big] + \frac{e^{\pm i\pi\alpha}z^{-\alpha}}{\Gamma(\beta-\alpha)}\big[1+o(1)\big]$$

as $|z| \to \infty$ uniformly in any sector $-\pi/2 + \epsilon \leq \pm\arg z \leq 3\pi/2 - \epsilon$ with $\epsilon > 0$ for appropriate choices of branches for $z^{\alpha-\beta}$ and $z^{-\alpha}$ and an appropriate determination of $\arg z$ [2, eq. 13.7.2]. It follows that for any $\theta \in (0, \pi/2)$ there exists a constant $\mu < 1$ such that

$$\Gamma(\alpha)\,\mathbf{M}(\alpha,\beta,z) = \begin{cases} z^{\alpha-\beta}e^z\,[1+o(1)] & \text{for } |\arg z| \leq \theta, \\ O\Big(e^{\mu|z|}\Big) & \text{for } |\arg z| > \theta \end{cases}$$

as $|z| \to \infty$ uniformly in each of these sectors.

From the above information we deduce that the function $\mathbf{M}$ has a single direction of maximal exponential growth. In the notation of the asymptotic assumption on the functions f we have considered in this paper (see equation (1)) we have $a = \alpha - \beta$, $b = 0$, and $\lambda = 1$. Applying Theorems 5, 7, and 13 to these functions yields the following collection of results.

Theorem 19 *Let*

$$p_n[\mathbf{M}](z) = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n}\frac{\Gamma(k+\alpha)}{\Gamma(k+\beta)k!}z^k$$

denote the n^{th} *partial sum of the Maclaurin series for* $\mathbf{M}(\alpha,\beta,z)$ *and let*

$$S = \{z \in \mathbb{C} : |z\exp(1-z)| = 1,\ |z| \leq 1,\ \textit{and}\ \operatorname{Re} z > 0\}.$$

The limit points of the zeros of the scaled partial sums $p_{n-1}[\mathbf{M}](nz)$ *in the right half-plane are precisely the points of* S.

Let $\xi \in S$, $\xi \neq 1$ *and define*

$$\tau = \operatorname{Im}(\xi - 1 - \log\xi),$$

$$\tau_n \equiv \tau n \pmod{2\pi}, \quad -\pi < \tau_n \leq \pi,$$

and

$$z_n(w) = \xi\left(1 + \frac{\log n}{2(1-\xi)n} - \frac{w - i\tau_n}{(1-\xi)n}\right).$$

Then

$$\lim_{n\to\infty}\frac{p_{n-1}[\mathbf{M}](nz_n(w))}{\mathbf{M}(\alpha,\beta,nz_n(w))} = 1 - \frac{e^{-w}}{\xi^{\alpha-\beta}(1-\xi)\sqrt{2\pi}} \tag{79}$$

uniformly on compact subsets of the w*-plane.*

Additionally,

$$\lim_{n\to\infty} \frac{p_{n-1}[\mathbf{M}](n+w\sqrt{n})}{\mathbf{M}(\alpha,\beta,n+w\sqrt{n})} = \frac{1}{2}\,\mathrm{erfc}\left(\frac{w}{\sqrt{2}}\right) \tag{80}$$

uniformly on compact subsets of $\mathrm{Re}\,w < 0$.

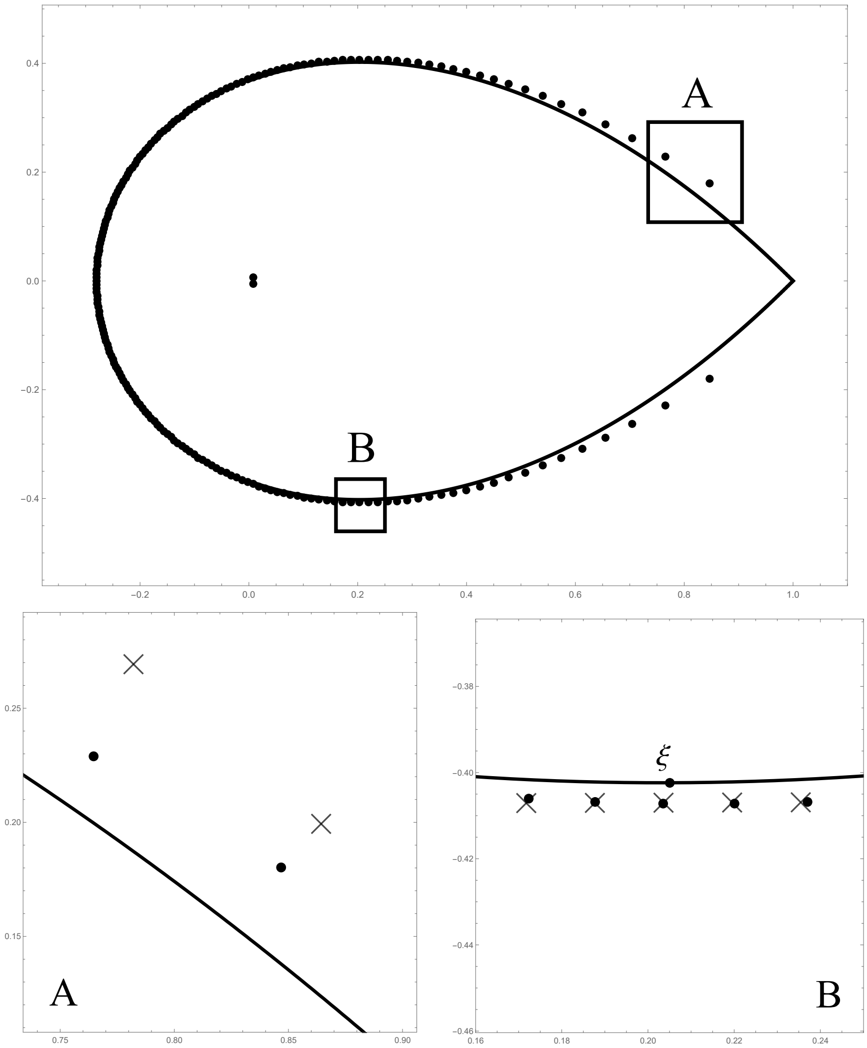

Fig. 6 The zeros of $p_{n-1}[\mathbf{M}](nz)$ for $\alpha = -1/2$, $\beta = -5/2$, and $n = 200$ and their limit curve in the right half-plane. Region A, magnified bottom-left, shows the approximations for the zeros in that region, represented as crosses, which are given by the corner scaling limit in (80). Region B, magnified bottom-right, shows the approximations in that region given by the curve scaling limit in (79) using the indicated reference point ξ.

7.4 Exponential integrals

Let $-1 \leq r < 1$ and let $g\colon [r,1] \to \mathbb{C} \cup \{\infty\}$ be a measurable, integrable function satisfying

$$g(t) = (t-r)^p g_1(t-r) = (1-t)^q g_2(1-t),$$

where

(1) $\operatorname{Re} p > -1$ and $\operatorname{Re} q > -1$,
(2) $g_1(0)$ and $g_2(0)$ are both finite and nonzero, and
(3) in a neighborhood of $t=0$, both $g_1'(t)$ and $g_2'(t)$ exist and are bounded.

Define the function

$$f(z) = \int_r^1 e^{zt} g(t)\, dt. \tag{81}$$

Under the above conditions this f is an entire function of order 1.

By Watson's lemma (see e.g. [11, Secs. 2.2 and 2.3]) the function f obeys the asymptotics

$$f(z) \sim g_2(0)\Gamma(q+1) z^{-q-1} e^z$$

and

$$f(-z) \sim g_1(0)\Gamma(p+1) z^{-p-1} e^{-rz}$$

as $|z| \to \infty$ uniformly in any sector $|\arg z| \leq \theta$ with $0 \leq \theta < \pi/2$. Additionally, if $|\arg z| \leq \pi/2 - \theta$ then

$$\begin{aligned} |f(\pm iz)| &\leq \int_r^1 e^{\mp t \operatorname{Im} z} |g(t)|\, dt \\ &= \int_r^1 e^{\mp |z| t \sin \arg z} |g(t)|\, dt \\ &\leq \int_r^1 e^{|zt| \cos\theta} |g(t)|\, dt \\ &\leq e^{|z|\cos\theta} \int_r^1 |g(t)|\, dt. \end{aligned}$$

It follows that for any fixed $\theta \in (0, \pi/2)$, if $-1 < r < 1$ then

$$[g_2(0)\Gamma(q+1)]^{-1} f(z) = \begin{cases} z^{-q-1} e^z \,[1+o(1)] & \text{for } |\arg z| \leq \theta, \\ O\Big(e^{\mu|z|}\Big) & \text{for } |\arg z| > \theta \end{cases} \tag{82}$$

and if $r = -1$ then

$$[g_2(0)\Gamma(q+1)]^{-1} f(z) = \begin{cases} z^{-q-1} e^z \,[1+o(1)] & \text{for } |\arg z| \leq \theta, \\ A(-z)^{-p-1} e^{-z} \,[1+o(1)] & \text{for } |\arg -z| \leq \theta, \\ O\Big(e^{\mu|z|}\Big) & \text{otherwise} \end{cases} \tag{83}$$

as $|z| \to \infty$ uniformly in each sector, where $\mu < 1$ and

$$A = \frac{g_1(0)\Gamma(p+1)}{g_2(0)\Gamma(q+1)}.$$

Applying Theorems 5, 7, 13, 6, 8, and 14 to this function yields the following collection of results.

Theorem 20 *Let*

$$p_n[f](z) = \sum_{k=0}^{n} \frac{z^k}{k!} \int_r^1 t^k g(t)\, dt$$

denote the n^{th} partial sum of the Maclaurin series for $f(z)$ and let

$$S = \{z \in \mathbb{C} : |z \exp(1-z)| = 1,\ |z| \le 1,\ \text{and } \operatorname{Re} z > 0\}.$$

If $-1 < r < 1$ then the limit points of the zeros of the scaled partial sums $p_{n-1}[f](nz)$ in the right half-plane are precisely the points of S. If $r = -1$, the limit points which do not lie on the imaginary axis are precisely the points of the set $S \cup -S$.

Let $\xi \in S$, $\xi \ne 1$ and define

$$\tau = \operatorname{Im}(\xi - 1 - \log \xi).$$

Define the sequence τ_n by the conditions

$$\tau n \equiv \tau_n \pmod{2\pi}, \qquad -\pi < \tau_n \le \pi$$

and let

$$z_n^1(w) = \xi \left(1 + \frac{\log n}{2(1-\xi)n} - \frac{w - i\tau_n}{(1-\xi)n}\right).$$

Define the sequence σ_n by the conditions

$$\pi n \equiv \sigma_n \pmod{2\pi}, \qquad -\pi < \sigma_n \le \pi$$

and let

$$z_n^2(w) = \xi \left[1 + \left(p - q + \frac{1}{2}\right) \frac{\log n}{(1-\xi)n} - \frac{w - i\sigma_n - i\tau_n}{(1-\xi)n}\right].$$

If $-1 < r < 1$ or if $r = -1$ and $\operatorname{Re} q < \operatorname{Re} p$ then

$$\lim_{n\to\infty} \frac{p_{n-1}[f](nz_n^1(w))}{f(nz_n^1(w))} = 1 - \frac{\xi^{q+1} e^{-w}}{(1-\xi)\sqrt{2\pi}} \tag{84}$$

as $n \to \infty$ uniformly on compact subsets of the w-plane. When $r = -1$, if $\operatorname{Re} q > \operatorname{Re} p$ then

$$\lim_{n\to\infty} \frac{p_{n-1}[f](nz_n^2(w))}{f(nz_n^2(w))} = 1 - \frac{A\xi^{q+1} e^{-w}}{(1+\xi)\sqrt{2\pi}} \tag{85}$$

and if $\operatorname{Re} q = \operatorname{Re} p$ then

$$\frac{p_{n-1}[f](nz_n^1(w))}{f(nz_n^1(w))} = 1 - \left(\frac{1}{1-\xi} + \frac{A(-1)^n n^{q-p}}{1+\xi}\right) \frac{\xi^{q+1} e^{-w}}{\sqrt{2\pi}} + o(1)$$

as $n \to \infty$. Both of these limits are uniform with respect to w on compact subsets of the w-plane.

If $-1 < r < 1$ or if $r = -1$ and $\operatorname{Re} q < \operatorname{Re} p + \frac{1}{2}$ then

$$\lim_{n\to\infty} \frac{p_{n-1}[f](n + w\sqrt{n})}{f(n + w\sqrt{n})} = \frac{1}{2} \operatorname{erfc}\left(\frac{w}{\sqrt{2}}\right) \tag{86}$$

uniformly on compact subsets of $\operatorname{Re} w < 0$. *If* $r = -1$ *and* $\operatorname{Re} q = \operatorname{Re} p + \frac{1}{2}$ *then*

$$\frac{p_{n-1}[f](n + w\sqrt{n})}{f(n + w\sqrt{n})} = \frac{1}{2}\operatorname{erfc}\left(\frac{w}{\sqrt{2}}\right) - \frac{A(-1)^n n^{q-p-1/2} e^{-w^2/2}}{2\sqrt{2\pi}} + o(1)$$

as $n \to \infty$ *uniformly on compact subsets of* $\operatorname{Re} w < 0$.

Let

$$z_n^3(w) = 1 - \sqrt{\frac{\kappa \log n}{n}} - \frac{w - i\sigma_n}{\sqrt{\kappa n \log n}},$$

where

$$\kappa := 2(q - p) - 1$$

and the principal branch of the square root is taken.

If $r = -1$ *and* $\operatorname{Re} q > \operatorname{Re} p + \frac{1}{2}$ *then*

$$\lim_{n\to\infty} \frac{p_{n-1}[f](nz_n^3(w))}{f(nz_n^3(w))} = 1 - \frac{Ae^{-w}}{2\sqrt{2\pi}} \tag{87}$$

uniformly on compact subsets of the w*-plane.*

7.5 Airy functions

The Airy functions of the first and second kind are entire functions defined for $z \in \mathbb{C}$ by the integrals

$$\operatorname{Ai}(z) = \frac{1}{2\pi i} \int_{e^{-i\pi/3}\infty}^{e^{i\pi/3}\infty} \exp\left(\frac{1}{3}t^3 - zt\right) dt$$

and

$$\operatorname{Bi}(z) = \frac{1}{2\pi} \int_{-\infty}^{e^{i\pi/3}\infty} \exp\left(\frac{1}{3}t^3 - zt\right) dt + \frac{1}{2\pi} \int_{-\infty}^{e^{-i\pi/3}\infty} \exp\left(\frac{1}{3}t^3 - zt\right) dt,$$

respectively.

The DLMF gives the following uniform asymptotics for Ai [2, eqns. 9.7.5 and 9.7.9]: for any fixed $\epsilon > 0$,

$$\operatorname{Ai}(z) \sim \frac{z^{-1/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3}z^{3/2}\right)$$

as $|z| \to \infty$ with $|\arg z| \le \pi - \epsilon$ and

$$\operatorname{Ai}(-z) = \frac{z^{-1/4}}{\sqrt{\pi}} \left[\cos\omega\,[1 + o(1)] + O\left(\frac{\sin\omega}{\omega}\right)\right]$$

as $|z| \to \infty$ with $|\arg z| \le 2\pi/3 - \epsilon$, where

$$\omega = \frac{2}{3}z^{3/2} - \frac{\pi}{4}.$$

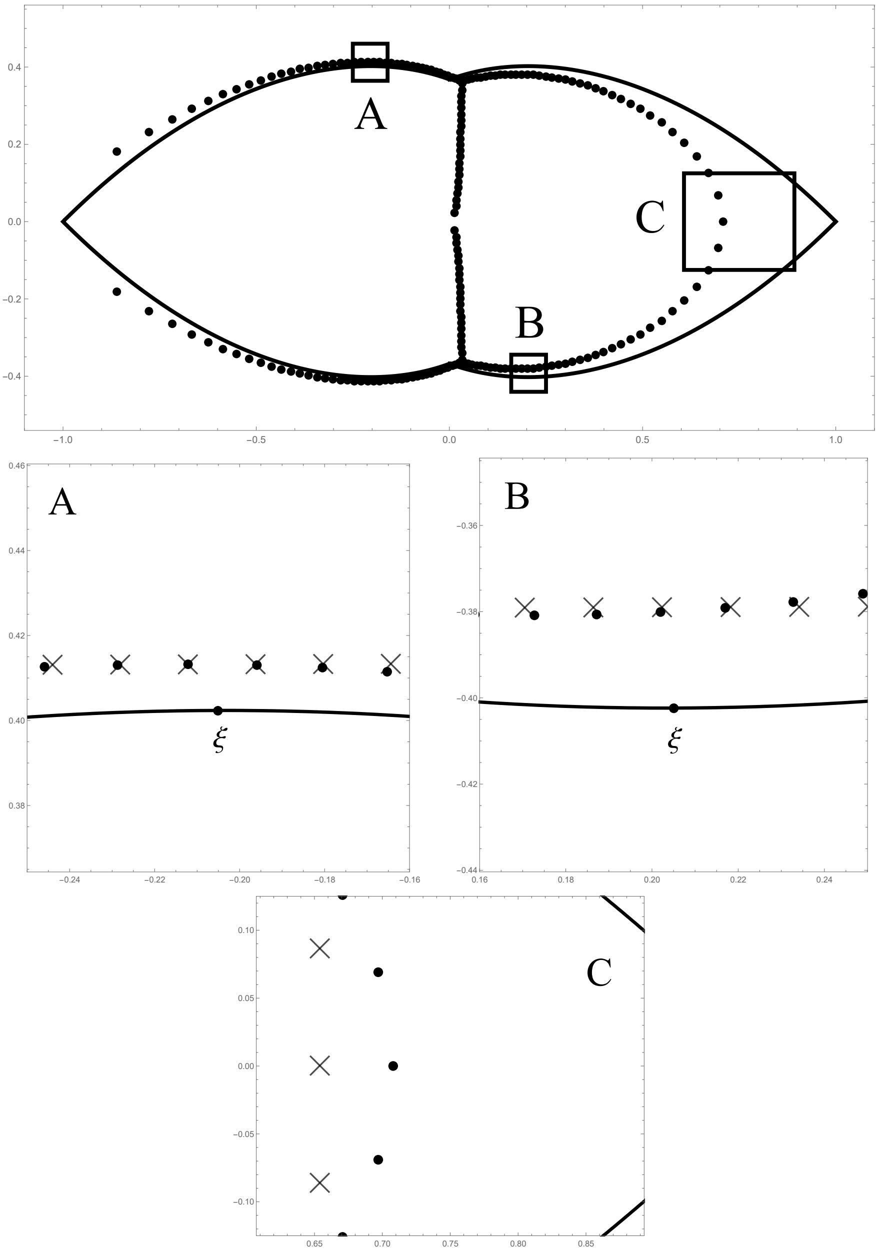

Fig. 7 The zeros of $p_{n-1}[f](nz)$ for $g(t) = (1-t)^3$, $r = -1$, and $n = 200$ and their limit curve. The magnified regions A, B, and C show the approximations for the zeros in those regions, represented as black crosses, given by the scaling limits in (84), (85), and (87), respectively. Note: to apply limit (84) we applied Theorem 20 to $f(-z)$. The points ξ used in the theorem are indicated.

It follows that for any $\epsilon > 0$ there is some $\mu < 2/3$ such that

$$\operatorname{Ai}(z) = \frac{1}{2\sqrt{\pi}} \times \begin{cases} z^{-1/4} \exp\left(-\frac{2}{3} z^{3/2}\right) [1 + o(1)] & \text{for } |\arg z| \leq \pi - \epsilon, \\ O\left(\exp\left(\mu |z|^{3/2}\right)\right) & \text{for } |\arg z| \geq \pi - \epsilon \end{cases}$$

as $|z| \to \infty$ uniformly in each sector. The quantity $\operatorname{Re}(-z^{3/2})$ is maximal on the rays $\arg z = \pm 2\pi/3$, so we conclude that Ai has two directions of maximal exponential growth.

We could now apply Theorems 6, 8, and 14 directly. However, in each of these results we restricted ourselves to sectors which are symmetric about the rays of maximal growth of the function. This was done to simplify the notation, and not because some aspect of the analysis requires it. Without additional comment we will use slight generalizations of the results in this direction. In particular, we will consider the function $\operatorname{Ai}(z)$ to have two half-open sectors of maximal growth, one $0 \leq \arg z < \pi - \epsilon$ and the other $-\pi + \epsilon < \arg z \leq 0$, even though the rays of maximal growth ($\arg z = 2\pi/3$ and $-2\pi/3$, respectively) do not bisect these sectors. With this in mind, the appropriately extended versions of Theorems 6, 8, and 14 give the following collection of results.

Theorem 21 *Let $p_n[\operatorname{Ai}](z)$ denote the n^{th} partial sum of the Maclaurin series for $\operatorname{Ai}(z)$, let*

$$S = \left\{ z \in \mathbb{C} : \left| z^{3/2} \exp\left(1 - z^{3/2}\right) \right| = 1,\ |z| \leq 1,\ \text{and } -2\pi/3 \leq \arg z < \pi/3 \right\},$$

and define

$$S_+ = S, \qquad S_- = \overline{S},$$

and

$$r_n = \left(\frac{2n}{3}\right)^{2/3}.$$

The limit points of the zeros of the scaled partial sums $p_{n-1}[\operatorname{Ai}]((3/2)^{2/3} r_n z)$ which do not lie on the ray $\arg z = \pi$ are precisely the points of the set

$$\left[(3/2)^{2/3} e^{i2\pi/3} S_+\right] \cup \left[(3/2)^{2/3} e^{-i2\pi/3} S_-\right].$$

Let $\xi \in S_\pm$, $\xi \neq 1$ and define

$$\tau = \operatorname{Im}\left(\xi^{3/2} - 1 - \frac{3}{2}\log \xi\right),$$

$$\tau_n \equiv \frac{\tau n}{3/2} \pmod{2\pi}, \quad -\pi < \tau_n \leq \pi,$$

and

$$z_n(w) = \xi \left(1 + \frac{\log n}{2(1 - \xi^{3/2})n} - \frac{w - i\tau_n}{(1 - \xi^{3/2})n}\right).$$

Then

$$\frac{p_{n-1}[\operatorname{Ai}]\left((3/2)^{2/3} e^{\pm i2\pi/3} r_n z_n(w)\right)}{\operatorname{Ai}\left((3/2)^{2/3} e^{\pm i2\pi/3} r_n z_n(w)\right)} = 1 - \left(\frac{1}{1-\xi} - \frac{e^{\mp i2\pi n/3}}{e^{\pm i2\pi/3} - \xi}\right) \frac{\xi^{1/4} e^{-w}}{\sqrt{3\pi}} + o(1) \tag{88}$$

as $n \to \infty$ uniformly on compact subsets of the w-plane.

Additionally,

$$\lim_{n\to\infty} \frac{p_{n-1}[\mathrm{Ai}]\big((3/2)^{2/3}e^{\pm i2\pi/3}r_n(1+w/\sqrt{n})\big)}{\mathrm{Ai}\big((3/2)^{2/3}e^{\pm i2\pi/3}r_n(1+w/\sqrt{n})\big)} = \frac{1}{2}\,\mathrm{erfc}\Big(w\sqrt{3/4}\Big) \tag{89}$$

uniformly on compact subsets of $\mathrm{Re}\, w < 0$.

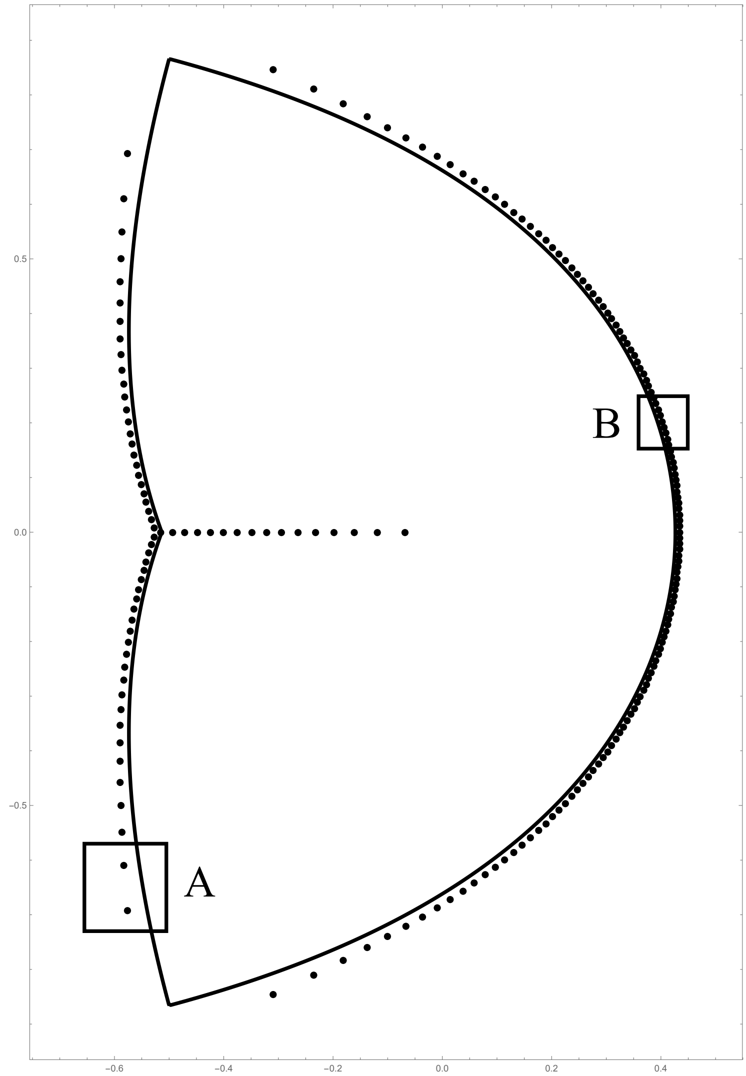

Fig. 8 The zeros of $p_{n-1}[\mathrm{Ai}]((3/2)^{2/3}r_n z)$ for $n = 200$ and their limit curve.

Remark 13 Due to the appearance of $e^{\pm i2\pi n/3}$ in the scaling limit corresponding to the arcs of the limit curve, the ratio $p_{n-1}[\mathrm{Ai}](\cdots)/\,\mathrm{Ai}(\cdots)$ may tend to one of

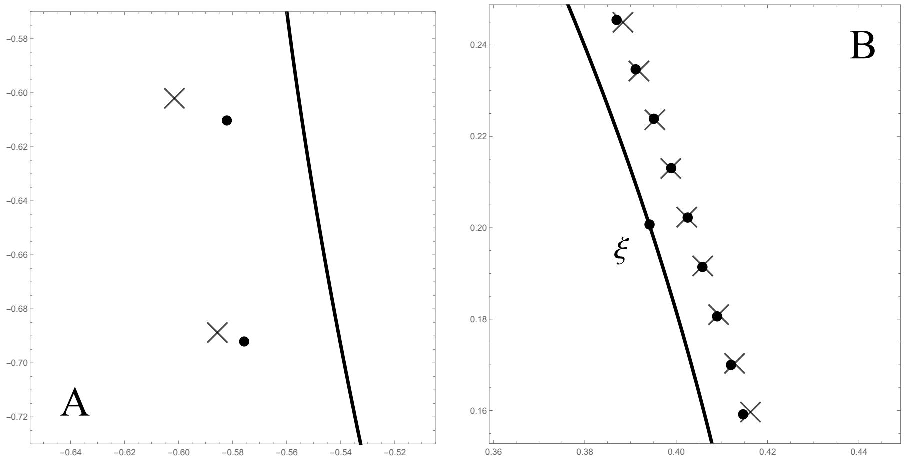

Fig. 9 Magnifications of the regions labeled in Figure 8. Region A shows the approximations for the zeros in that region, represented as crosses, which are given by the corner scaling limit in (89). Region B shows the approximations in that region given by the curve scaling limit in (88) using the indicated reference point ξ.

three different limits if we restrict n to run through any one of the three residue classes modulo 3.

The DLMF gives the following uniform asymptotics for Bi [2, eqns. 9.7.7, 9.7.11, and 9.7.13]: for any fixed $\epsilon > 0$,

$$\operatorname{Bi}(z) \sim \frac{z^{-1/4}}{\sqrt{\pi}} \exp\left(\frac{2}{3} z^{3/2}\right)$$

as $|z| \to \infty$ with $|\arg z| \leq \pi/3 - \epsilon$,

$$\operatorname{Bi}(-z) = \frac{z^{-1/4}}{\sqrt{\pi}} \left[-\sin\omega\big[1 + o(1)\big] + O\Big(\frac{\cos\omega}{\omega}\Big)\right]$$

as $|z| \to \infty$ with $|\arg z| \leq 2\pi/3 - \epsilon$, and

$$\operatorname{Bi}\Big(e^{\pm i\pi/3} z\Big) = e^{\pm i\pi/6} z^{-1/4} \sqrt{\frac{2}{\pi}} \left[\cos\left(\omega \mp \frac{i}{2}\log 2\right)\big[1 + o(1)\big] + O\left(\frac{\sin\big(\omega \mp \frac{i}{2}\log 2\big)}{\omega}\right)\right]$$

as $|z| \to \infty$ with $|\arg z| \leq 2\pi/3 - \epsilon$, where

$$\omega = \frac{2}{3} z^{3/2} - \frac{\pi}{4}.$$

It follows that for any $\epsilon > 0$ there is some $\mu < 2/3$ such that

$$\sqrt{\pi}\operatorname{Bi}(z) \sim \begin{cases} z^{-1/4} \exp\Big(\frac{2}{3} z^{3/2}\Big) & \text{for } |\arg z| \leq \pi/3 - \epsilon, \\ \frac{1}{2} e^{\mp i\pi/3} \Big(e^{\pm i2\pi/3} z\Big)^{-1/4} \exp\left[\frac{2}{3}\Big(e^{\pm i2\pi/3} z\Big)^{3/2}\right] & \text{for } \left|\arg e^{\pm i2\pi/3} z\right| \leq \pi/3 - \epsilon, \\ O\Big(\exp\Big(\mu |z|^{3/2}\Big)\Big) & \text{otherwise} \end{cases}$$

as $|z| \to \infty$ uniformly in each sector. Evidently the Bi function has three directions of maximal exponential growth. The discussion in Subsection 4.3 and Theorems 11 and 15 yield the following collection of results.

Theorem 22 *Let* $p_n[\mathrm{Bi}](z)$ *denote the* n^{th} *partial sum of the Maclaurin series for* $\mathrm{Bi}(z)$*, let*

$$S = \left\{ z \in \mathbb{C} : \left| z^{3/2} \exp\left(1 - z^{3/2}\right) \right| = 1,\ |z| \le 1,\ \textit{and}\ |\arg z| < \pi/3 \right\},$$

and define

$$r_n = \left(\frac{2n}{3} \right)^{2/3}.$$

The limit points of the zeros of the scaled partial sums $p_{n-1}[\mathrm{Bi}]((3/2)^{2/3} r_n z)$ *which do not lie on the rays* $\arg z = \pm\pi/3, \pi$ *are precisely the points of the set*

$$S \cup e^{i2\pi/3} S \cup e^{-i2\pi/3} S.$$

Let $\xi \in S$*,* $\xi \ne 1$ *and define*

$$\tau = \mathrm{Im}\left(\xi^{3/2} - 1 - \frac{3}{2} \log \xi \right),$$

$$\tau_n \equiv \frac{\tau n}{3/2} \pmod{2\pi}, \quad -\pi < \tau_n \le \pi,$$

and

$$z_n(w) = \xi \left(1 + \frac{\log n}{2(1 - \xi^{3/2})n} - \frac{w - i\tau_n}{(1 - \xi^{3/2})n} \right).$$

Then

$$\begin{aligned} &\frac{p_{n-1}[\mathrm{Bi}]\left((3/2)^{2/3} r_n z_n(w)\right)}{\mathrm{Bi}\left((3/2)^{2/3} r_n z_n(w)\right)} \\ &\quad = 1 - \left(\frac{1}{1-\xi} - \frac{e^{-i2\pi n/3}}{2(e^{i2\pi/3} - \xi)} - \frac{e^{i2\pi n/3}}{2(e^{-i2\pi/3} - \xi)} \right) \frac{\xi^{1/4} e^{-w}}{\sqrt{3\pi}} + o(1) \end{aligned}$$

and

$$\begin{aligned} &\frac{p_{n-1}[\mathrm{Bi}]\left((3/2)^{2/3} e^{\pm i2\pi/3} r_n z_n(w)\right)}{\mathrm{Bi}\left((3/2)^{2/3} e^{\pm i2\pi/3} r_n z_n(w)\right)} \\ &\quad = 1 - \left(\frac{1}{1-\xi} - \frac{2e^{\pm i2\pi n/3}}{e^{\mp i2\pi/3} - \xi} + \frac{e^{\mp i2\pi n/3}}{e^{\pm i2\pi/3} - \xi} \right) \frac{\xi^{1/4} e^{-w}}{\sqrt{3\pi}} + o(1) \end{aligned} \tag{90}$$

as $n \to \infty$ *uniformly on compact subsets of the* w*-plane.*

Additionally, for fixed $k \in \{-1, 0, 1\}$

$$\frac{p_{n-1}[\mathrm{Bi}]\left((3/2)^{2/3} e^{i2\pi k/3} r_n z_n(w)\right)}{\mathrm{Bi}\left((3/2)^{2/3} e^{i2\pi k/3} r_n z_n(w)\right)} = \frac{1}{2} \operatorname{erfc}\left(w\sqrt{3/4} \right) \tag{91}$$

uniformly on compact subsets of $\mathrm{Re}\, w < 0$.

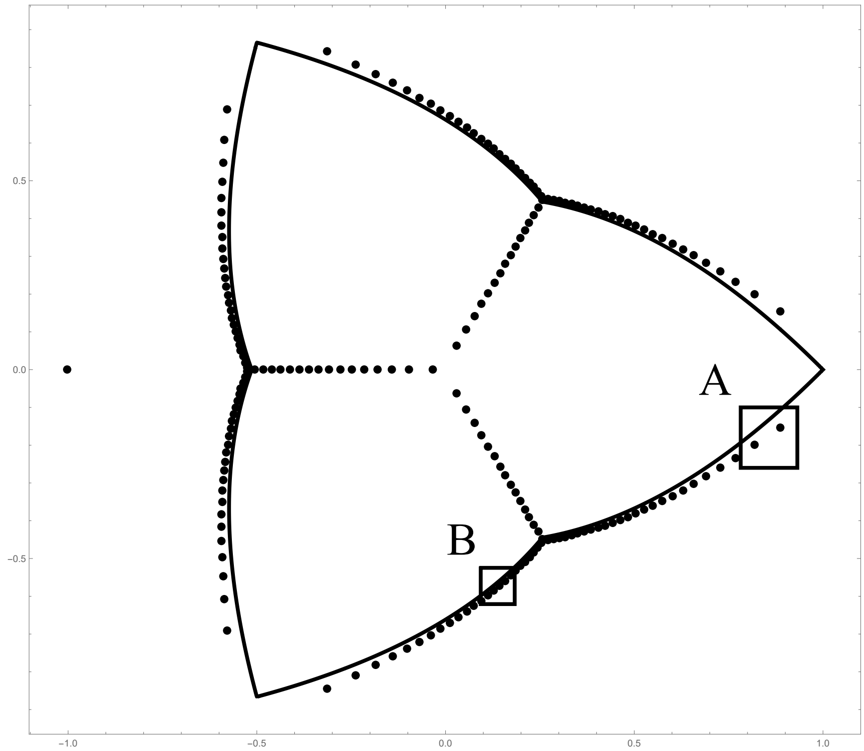

Fig. 10 The zeros of $p_{n-1}[\mathrm{Bi}]((3/2)^{2/3} r_n z)$ for $n = 200$ and their limit curve.

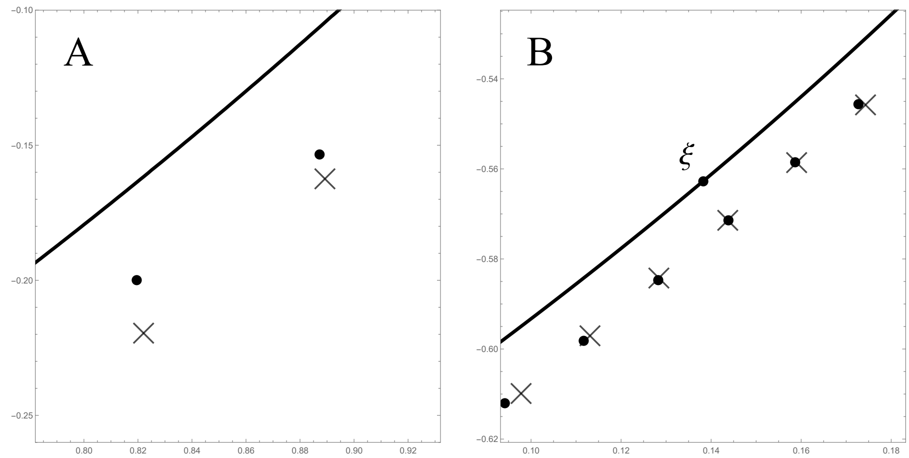

Fig. 11 Magnifications of the regions labeled in Figure 10. Region A shows the approximations for the zeros in that region, represented as crosses, which are given by the corner scaling limit in (91). Region B shows the approximations in that region given by the curve scaling limit in (90) using the indicated reference point ξ.

7.6 Parabolic cylinder functions

The parabolic cylinder function $U(a,z)$ is an entire function which is defined for $a,z \in \mathbb{C}$ as the solution of the differential equation

$$\frac{d^2u}{dz^2} - \left(\frac{1}{4}z^2 + a\right)u = 0$$

identified by the asymptotic behavior

$$U(a,z) \sim z^{-a-1/2}e^{-z^2/4}$$

as $|z| \to \infty$ with $|\arg z| \le 3\pi/4 - \epsilon$ and

$$U(a,z) = z^{-a-1/2}e^{-z^2/4}\left[1+o(1)\right] \pm \frac{i\sqrt{2\pi}}{\Gamma(a+1/2)}e^{\mp i\pi a}z^{a-1/2}e^{z^2/4}\left[1+o(1)\right]$$

as $|z| \to \infty$ with $\pi/4+\epsilon \le \pm\arg z \le 5\pi/4-\epsilon$, where in the latter appropriate choices for the branches of $z^{-a-1/2}$ and $z^{a-1/2}$ and an appropriate determination of arg is made [2, eqns. 12.9.1 and 12.9.3]. These functions are sometimes written using the alternate notation

$$D_\nu(z) = U(-\nu-1/2, z),$$

as in [1, sec. 19.3].

From the above asymptotic it follows that U has three directions of maximal exponential growth. Indeed, for any $\epsilon > 0$ there is a $\mu < 1/4$ such that

$$U(a,z) \sim \begin{cases} \frac{\sqrt{2\pi}}{\Gamma(a+1/2)}(-z)^{a-1/2}\exp\left[\frac{1}{4}(-z)^2\right] & \text{for } |\arg -z| \le \pi/4-\epsilon, \\ e^{i\pi(a/2+1/4)}(iz)^{-a-1/2}\exp\left[\frac{1}{4}(iz)^2\right] & \text{for } -\pi/4+\epsilon < \arg iz < \pi/2-\epsilon, \\ e^{-i\pi(a/2+1/4)}(z/i)^{-a-1/2}\exp\left[\frac{1}{4}(z/i)^2\right] & \text{for } -\pi/2+\epsilon < \arg(z/i) < \pi/4-\epsilon, \\ O\big(\exp(\mu|z|^2)\big) & \text{otherwise} \end{cases}$$

as $|z| \to \infty$ uniformly in each sector. As with the Ai function we will consider the half-open sectors $0 \le \pm\arg z < 3\pi/4$ to be maximal growth sectors even though the maximal growth rays $\arg z = \pm\pi/2$ do not bisect them. With this in mind, the discussion in Subsection 4.3 and Theorems 9, 10, 11, 12 and 15 yield the following collection of results.

Theorem 23 *Let $p_n[U](z)$ denote the n^{th} partial sum of the Maclaurin series for $U(a,z)$, let*

$$S = \left\{z \in \mathbb{C} : \left|z^2\exp\Big(1-z^2\Big)\right| = 1,\ |z| \le 1,\ \textit{and}\ -\pi/2 \le \arg z < \pi/4\right\},$$

and define

$$S_{\pi/4} = S \cap \{z \in \mathbb{C} : |\arg z| < \pi/4\}, \qquad S_+ = S, \qquad S_- = \overline{S}$$

and

$$r_n = \left(\frac{n}{2}\right)^{1/2}.$$

The limit points of the zeros of the scaled partial sums $p_{n-1}[U](2r_n z)$ *which do not lie on the rays* $\arg z = \pm 3\pi/4$ *are precisely the points of the set*

$$iS_+ \cup -iS_- \cup -S_{\pi/4}.$$

Let $\xi \in S_{\pi/4}$, $\xi \neq 1$, *and define*

$$\tau = \operatorname{Im}\Bigl(\xi^2 - 1 - 2\log\xi\Bigr),$$

$$\tau_n \equiv \frac{\tau n}{2} \pmod{2\pi}, \quad -\pi < \tau_n \leq \pi,$$

$$\sigma_n \equiv \frac{\pi n}{2} \pmod{2\pi}, \quad -\pi < \sigma_n \leq \pi,$$

$$z_n^1(w) = \xi\left(1 + \frac{\log n}{2(1-\xi^2)n} - \frac{w - i\tau_n}{(1-\xi^2)n}\right),$$

$$z_n^2(w) = \xi\left(1 + \frac{(2a+1)\log n}{2(1-\xi^2)n} - \frac{w - i\sigma_n - i\tau_n}{(1-\xi^2)n}\right),$$

and

$$z_n^3(w) = 1 - \sqrt{\frac{\kappa\log n}{n}} - \frac{w - i\sigma_n}{2\sqrt{\kappa n\log n}},$$

where $\kappa = -(2a+1)/2$.

If $\operatorname{Re} a > 0$ *then*

$$\lim_{n\to\infty} \frac{p_{n-1}[U](-2r_n z_n^1(w))}{U(a, -2r_n z_n^1(w))} = 1 - \frac{\xi^{1/2-a}e^{-w}}{2\sqrt{\pi}(1-\xi)},$$

if $\operatorname{Re} a = 0$ *then*

$$\begin{aligned}
&\frac{p_{n-1}[U](-2r_n z_n^1(w))}{U(a, -2r_n z_n^1(w))} \\
&\quad = 1 - \left[\frac{1}{1-\xi} + \frac{\Gamma(a+1/2)r_n^{-2a}}{4^a\sqrt{2\pi}}\left(\frac{i^{1-n}\exp\{i\pi(a/2+1/4)\}}{i-\xi}\right.\right. \\
&\qquad\qquad \left.\left. - \frac{(-i)^{1-n}\exp\{-i\pi(a/2+1/4)\}}{i+\xi}\right)\right]\frac{e^{-w}}{2\sqrt{\pi}\xi^{a-1/2}} + o(1)
\end{aligned}$$

as $n \to \infty$, *and if* $\operatorname{Re} a < 0$ *then*

$$\begin{aligned}
&\frac{p_{n-1}[U](-2r_n z_n^2(w))}{U(a, -2r_n z_n^2(w))} \\
&\quad = 1 - \left(\frac{\exp\{i\pi(a/2+3/4)\}}{i-\xi} + \frac{(-1)^n\exp\{-i\pi(a/2-1/4)\}}{i+\xi}\right)\frac{\Gamma(a+1/2)e^{-w}}{2^{a+3/2}\pi\xi^{a-1/2}} \\
&\qquad + o(1) \qquad (92)
\end{aligned}$$

as $n \to \infty$, *with each limit holding uniformly on compact subsets of the* w*-plane.*

If $\operatorname{Re} a > -1/2$ *then*

$$\lim_{n\to\infty} \frac{p_{n-1}[U](-2r_n(1+w/\sqrt{n}))}{U(a, -2r_n(1+w/\sqrt{n}))} = \frac{1}{2}\operatorname{erfc}(w)$$

and if $\operatorname{Re} a = -1/2$ *then*

$$\begin{aligned}&\frac{p_{n-1}[U](-2r_n(1+w/\sqrt{n}))}{U(a,-2r_n(1+w/\sqrt{n}))}\\&\qquad=\frac{1}{2}\operatorname{erfc}(w)-\Big((-i)^n e^{i\pi a/2}+i^n e^{-i\pi a/2}\Big)\frac{\Gamma(a+1/2)e^{-w^2}}{4^{a+1}\pi\sqrt{2}}+o(1)\end{aligned}$$

as $n \to \infty$*, with each limit holding uniformly on compact subsets of* $\operatorname{Re} w < 0$*. If* $\operatorname{Re} a < -1/2$ *then*

$$\frac{p_{n-1}[U](-2r_n z_n^3(w))}{U(a,-2r_n z_n^3(w))}=1-\Big(e^{i\pi a/2}+(-1)^n e^{-i\pi a/2}\Big)\frac{\Gamma(a+1/2)e^{-w}}{2^{a+2}\pi}+o(1) \quad (93)$$

as $n \to \infty$ *uniformly on compact subsets of the* w*-plane.*

Remark 14 The parabolic cylinder function has three directions of maximal exponential growth ($z \to -\infty$ and $z \to \pm i\infty$), each of which have different arc and corner scaling limits depending on the value of $\operatorname{Re} a$. In total there are 12 different scaling limits that can be obtained from the results in this paper. For example, when $\operatorname{Re} a < 1/2$ we have

$$\lim_{n\to\infty}\frac{p_{n-1}[U](\pm 2ir_n(1+w/\sqrt{n}))}{U(a,\pm 2ir_n(1+w/\sqrt{n}))}=\frac{1}{2}\operatorname{erfc}(w) \quad (94)$$

uniformly on compact subsets of $\operatorname{Re} w < 0$. Rather than listing all of these limits in Theorem 23 we have focused only on the ones relating to the direction $z \to -\infty$.

8 Conclusion

In his 1944 thesis [19] (see also [20]), Rosenbloom considered arbitrary entire functions with positive, finite order defined by power series

$$f(z) := \sum_{k=0}^{\infty} a_k z^k$$

and showed that, if some subsequence of $f(|a_n|^{-1/n}z)^{1/n}$ converges to an analytic function $g(z)$ on some domain X, then the zeros of the subsequence of scaled partial sums $p_n[f](|a_n|^{-1/n}z)$ converge to the curve $|g(z)/z| = 1$ in X. For example, Rosenbloom tells us that if

$$\lim_{n\to\infty} f(|a_n|^{-1/n}ez)^{1/n} = e^z \quad (95)$$

in some domain then the zeros of the scaled partial sums $p_n[f](|a_n|^{-1/n}ez)$ converge precisely to the classical Szegő curve $|ze^{1-z}| = 1$ in that domain.

These results can be thought of as the “first-order” theory for the zeros of partial sums of power series for entire functions. Given the above information about the limiting behavior of $f(|a_n|^{-1/n}z)^{1/n}$, Rosenbloom deduces the limiting behavior of the zeros of the scaled partial sums $p_n[f](|a_n|^{-1/n}z)$.

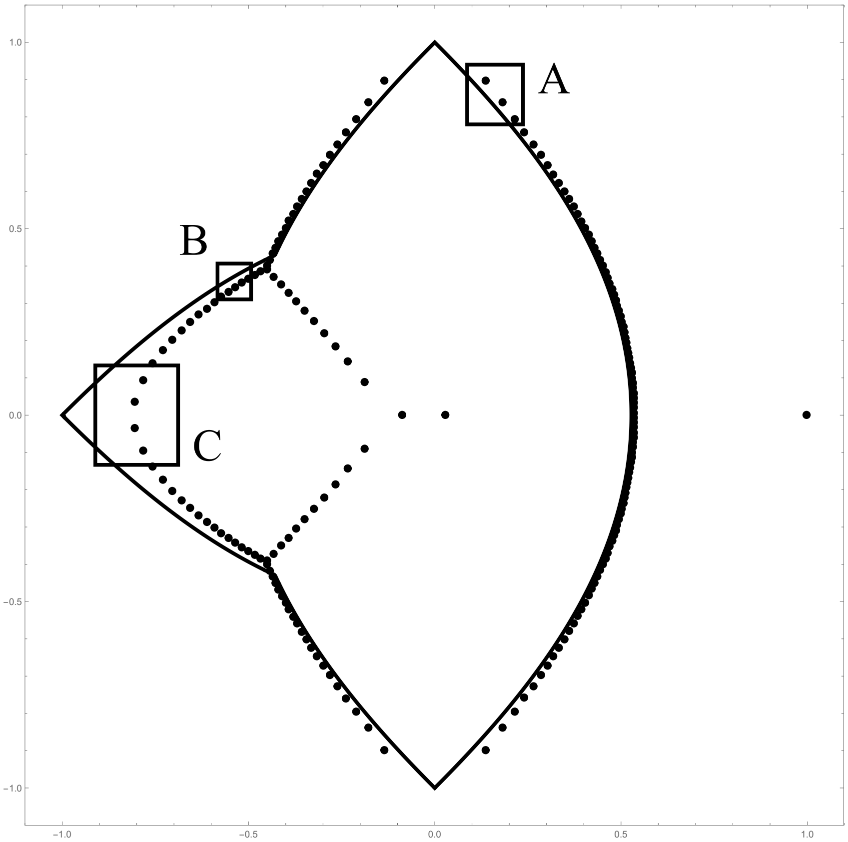

Fig. 12 The zeros of $p_{n-1}[U](2r_n z)$ for $a = -2$ and $n = 200$ and their limit curve.

The assumptions made in this paper are stronger than Rosenbloom's. Whereas Rosenbloom would only assume something like (95), we would instead assume that

$$f(nz) \sim b_n z^a e^{nz} \tag{96}$$

as $n \to \infty$ for z in some domain for some sequence (b_n) of complex numbers. Rosenbloom-type results can indeed be deduced from this—i.e. the proper scaling of the zeros (they will scale like n), as well as the fact that the zeros of the scaled partial sums $p_{n-1}[f](nz)$ will accumulate on the portion of the classical Szegő curve $|ze^{1-z}| = 1$ in the relevant domain. However, under this stronger assumption (96) we can go one step further than Rosenbloom and deduce not only how quickly the zeros approach the limit curve but also information about their geometry as they do so. In this way the results in this paper can be considered part of the "second-order" theory of the zeros.

The scaling limits in Section 5 tell us that the scaled partial sums of the entire functions we consider have zeros which approach their limit curve at a rate of approximately $\log n/n$, which are (locally) separated from each other by a distance of approximately $1/n$, and which approximately lie on straight lines

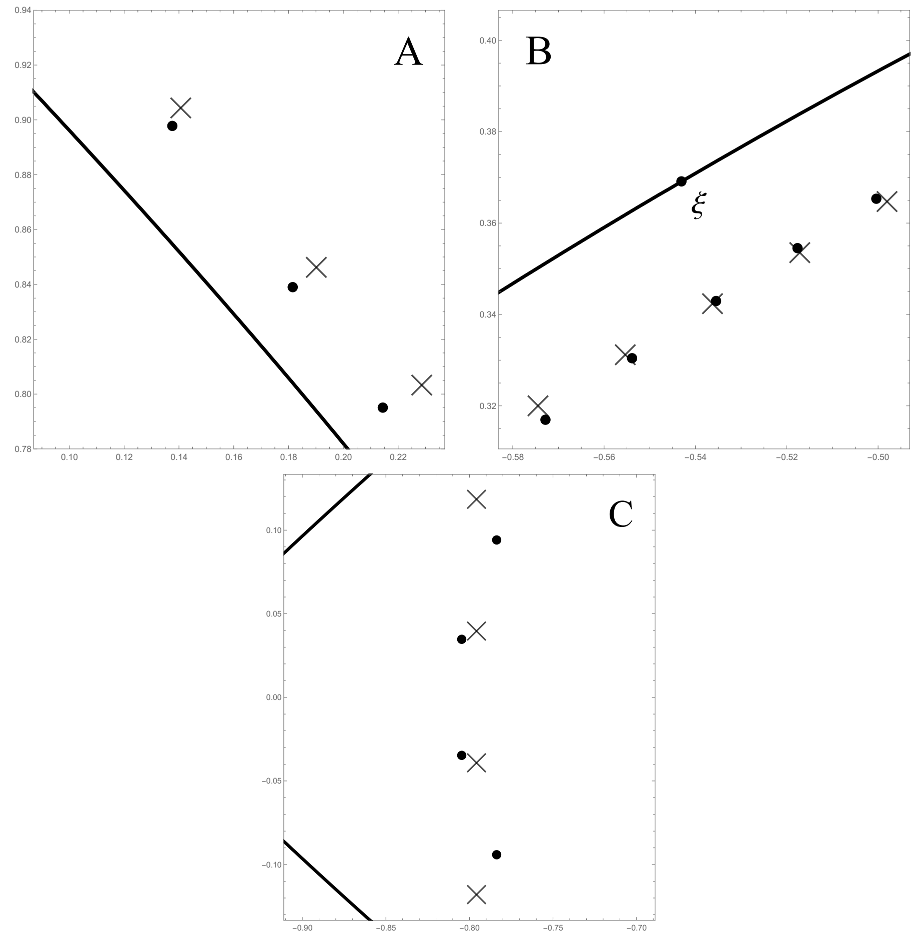

Fig. 13 Magnifications of the regions labeled in Figure 12. Region A shows the approximations for the zeros in that region, represented as crosses, which are given by the corner scaling limit in (94). Region B shows the approximations in that region given by the curve scaling limit in (92) using the indicated reference point ξ. Region C shows the approximations in that region given by the corner scaling limit in (93).

parallel to their limit curve. Similarly, the scaling limits in Section 6 tell us that the scaled partial sums have zeros which either

- approach the convex corner of their limit curve at a rate of approximately $1/\sqrt{n}$, are (locally) separated from each other by a distance of approximately $1/\sqrt{n}$, and approximately lie on rays with arguments $\pm 3\pi/4$ originating at the corner of the curve, or
- approach the convex corner of their limit curve at a rate of approximately $\sqrt{(\log n)/n}$, are (locally) separated from each other by a distance of approximately $1/\sqrt{n \log n}$, and approximately lie on a straight line perpendicular to the ray from the origin through the corner of the curve.

The details of these facts are given in Subsections 5.4 and 6.4.

This second-order information is detailed enough to imply the validity of the Saff-Varga Width Conjecture for these entire functions in their maximal exponential growth sectors. However, while substantial, this class of functions is smaller than the one considered by Rosenbloom. A verification of the Width Conjecture for Rosenbloom's class would mark a significant step forward in the theory. It would be interesting to see whether the ideas and techniques used here can be generalized to that case.

A proof of the Width Conjecture in the general setting still eludes us.

Acknowledgements In memory of my grandmother, Jacquelyn Gray Belzano.

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