A note about power series asymptotics

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Suppose we have a power series

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} L(k),$$

where L is a function satisfying

$$\lim_{x \to \infty} \frac{L(x)}{L(x+1)} = 1.$$

If we want to determine the behavior of $f(x)$ as $x \to \infty$, one way to proceed would be to determine which terms of the series contribute most to its size. If there is a peak term, it would occur approximately when

$$\frac{x^k}{k!} L(k) \approx \frac{x^{k+1}}{(k+1)!} L(k+1),$$

which is the same as

$$\frac{L(k)}{L(k+1)} \approx \frac{x}{k+1}.$$

But if k is large (this will be true for large x), then $L(k)/L(k+1) \approx 1$, so we have

$$k \approx x - 1 \approx x.$$

We thus expect that we can approximate the sum by approximating the slowly-varying factors of the summand near this peak at $k \approx x$:

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} L(k) \approx \sum_{k=0}^{\infty} \frac{x^k}{k!} L(x) = L(x) e^x.$$

In particular, this heuristic holds when $L(x) = \log x$ and $L(x) = 1/\sqrt{x}$, as in these two questions:

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