A Note on the Zeros of $\sum_{r=0}^{n} \frac{z^r}{r!} = 0$.

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In this note it is proposed to shew, by an elementary method, that the zeros of $\sum_{r=0}^{n} \frac{z^r}{r!}$, for large n, lie in the region $n > |z| > n/e^2$.

1. We shall first prove that for all n, the zeros all lie in $|z| < n$.

Let $f_n(z) = \sum_{r=0}^{n} \frac{z^r}{r!}$, and $\phi(z) = f_n(nz)$.

Let

$$g(z) = z^n \phi\left(\frac{1}{z}\right) = b_0 + b_1 z + \cdots + b_n z^n;$$

then

$$b_0 = \frac{n^n}{n!}, \quad b_1 = \frac{n^{n-1}}{(n-1)!}, \quad \ldots,$$

and, obviously,

$$b_0 = b_1 > \cdots > b_n.$$

Then all the zeros of $g(z)$ lie outside the unit-circle (Vide e.g. Landau: *Ergebnisse der Funktionentheorie*, p. 26, Satze von Eneström).

As the proof is short and simple we give it here:

$$(1 - z)g(z) = b_0 - \left\{(b_0 - b_1)z + (b_1 - b_2)z^2 + \cdots + (b_{n-1} - b_n)z^n + b_n z^{n+1}\right\}$$

Hence for $|z| \le 1$, excluding $z = 1$, we have

$$|(1 - z)g(z)| \ge b_0 - \{(b_0 - b_1) + (b_1 - b_2) + \cdots + (b_{n-1} - b_n) + b_n\} = 0.$$

It is obvious that the right side in the above expression for $(1 - z)g(z)$ can be zero only when $z = 1$. Hence $|g(z)| > 0$ when $|z| \le 1$.

Going back now to the function $f_n(z)$, we see immediately that all its roots lie in the region $|z| < n$.

2. At a zero, say z_0, of $f_n(z)$ we have

$$\frac{z_0^{n+1}}{(n+1)!} + \frac{z_0^{n+2}}{(n+2)!} + \cdots = e^{z_0}. \tag{2 \cdot 1}$$

Also $|z_0| < n$. Let $|z_0| = (n+1)u_0$.

Then

$$\begin{aligned} \exp\{-|z_0|\} &< \frac{|z_0|^{n+1}}{(n+1)!} + \cdots < \frac{|z_0|^{n+1}}{(n+1)!(1-u_0)} \\ &= \frac{[(n+1)u_0]^{n+1}}{[(n+1)/e]^{n+1}} \cdot \frac{1+\epsilon_n}{\sqrt{2\pi n}\cdot(1-u_0)} \qquad \text{for large } n. \end{aligned}$$

Hence

$$u_0 \cdot e \cdot (1+\epsilon_n)^{1/(n+1)} \div \{(1-u_0)\sqrt{2\pi n}\}^{1/(n+1)} > e^{-u_0}. \tag{2 \cdot 2}$$

Now by (§1) $u_0 < n/(n+1)$, so that $1 < \dfrac{1}{1-u_0} < n+1$ and $1 < (1-u_0)^{-1/(n+1)} < (n+1)^{1/(n+1)} = 1 + \epsilon'_n$.

Hence from $(2 \cdot 2)$ we have

$$u_0 e^{u_0+1} > 1 + \epsilon''_n$$

or

$$1 + u_0 + \log u_0 > \epsilon'''_n.$$

Now $1 + u_0 + \log u_0$ is a monotonic function which vanishes for one value u_0 in $\dfrac{1}{e} > u_0 > \dfrac{1}{e^2}$.

Hence when n is large $u_0 > \dfrac{1}{e^2}$ whence $n > |z_0| > \dfrac{n}{e^2}$ where z_0 is any root of $\displaystyle\sum_{r=0}^{n} \frac{z^r}{r!} = 0$.

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