Boundary asymptotics for power series

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1 Main results

Theorem 1. *Suppose* $a_n \sim b_n$ *as* $n \to \infty$*, where* b_n *is nonnegative or nonpositive for large* n*,* $\sum_{n \geq 0} b_n x^n$ *converges for all* $|x| < 1$*, and*

$$\sum_{n \geq 0} b_n = \pm\infty.$$

Then

$$\sum_{n \geq 0} a_n x^n \sim \sum_{n \geq 0} b_n x^n \qquad \text{as } x \to 1^-.$$

Proof. We can assume, without loss of generality, that b_n is eventually nonnegative. If necessary we just replace a_n and b_n with $-a_n$ and $-b_n$, respectively.

By assumption $a_n = b_n + b_n \delta_n$, where $\delta_n \to 0$ as $n \to \infty$. Fix $\epsilon > 0$, let $N \geq 0$ be such that $|\delta_n| < \epsilon$ and $b_n \geq 0$ for all $n > N$, then write

$$\begin{aligned}
\sum_{n \geq 0} a_n x^n &= \sum_{n \geq 0} (b_n + b_n \delta_n) x^n \\
&= \sum_{n \geq 0} b_n x^n + \sum_{n \geq 0} b_n \delta_n x^n \\
&= \sum_{n \geq 0} b_n x^n + \sum_{n > N} b_n \delta_n x^n + \sum_{n \leq N} b_n \delta_n x^n. \qquad (1)
\end{aligned}$$

We will show that the two rightmost sums are small when compared with $\sum_{n \geq 0} b_n x^n$.

Note that we can find an $X_0 \in [0, 1)$ such that $\sum_{n \geq 0} b_n x^n > 0$ for $X_0 < x < 1$. Throughout the proof we will assume that x lies in this interval.

First we'll handle the rightmost sum in (1). We have

$$\left| \frac{\sum_{n \leq N} b_n \delta_n x^n}{\sum_{n \geq 0} b_n x^n} \right| < \frac{\sum_{n \leq N} b_n |\delta_n|}{\sum_{n \geq 0} b_n x^n},$$

so since $\sum_{n\ge 0} b_n x^n \to \infty$ as $x \to 1^-$ we can find an $X_1 \in [0,1)$ such that

$$\left| \frac{\sum_{n\le N} b_n \delta_n x^n}{\sum_{n\ge 0} b_n x^n} \right| < \epsilon \tag{2}$$

for $X_1 < x < 1$.

Next we'll treat the sum $\sum_{n>N} b_n \delta_n x^n$. By our choice of N we may deduce that

$$\begin{aligned} \left| \sum_{n>N} b_n \delta_n x^n \right| &\le \sum_{n>N} b_n |\delta_n| x^n \\ &< \epsilon \sum_{n>N} b_n x^n \\ &= \epsilon \sum_{n\ge 0} b_n x^n - \epsilon \sum_{n\le N} b_n x^n \\ &< \epsilon \sum_{n\ge 0} b_n x^n + \epsilon \sum_{n\le N} |b_n|. \end{aligned} \tag{3}$$

We can find an $X_2 \in [0,1)$ such that

$$\frac{\sum_{n\le N} |b_n|}{\sum_{n\ge 0} b_n x^n} < 1$$

for $X_3 < x < 1$, so under this condition we gather from (3) that

$$\left| \sum_{n>N} b_n \delta_n x^n \right| < 2\epsilon \sum_{n\ge 0} b_n x^n. \tag{4}$$

In light of (2) and (4), if we assume that $\max\{X_0, X_1, X_2\} < x < 1$ then by (1) we have

$$\begin{aligned} \left| \frac{\sum_{n\ge 0} a_n x^n}{\sum_{n\ge 0} b_n x^n} - 1 \right| &= \left| \frac{\sum_{n\le N} b_n \delta_n x^n}{\sum_{n\ge 0} b_n x^n} + \frac{\sum_{n>N} b_n \delta_n x^n}{\sum_{n\ge 0} b_n x^n} \right| \\ &\le \left| \frac{\sum_{n\le N} b_n \delta_n x^n}{\sum_{n\ge 0} b_n x^n} \right| + \left| \frac{\sum_{n>N} b_n \delta_n x^n}{\sum_{n\ge 0} b_n x^n} \right| \\ &< 3\epsilon, \end{aligned}$$

which proves the result. $\square$

Corollary 2. *Suppose the sequence (a_n) has an asymptotic expansion given by*

$$a_n \approx \sum_{k\ge 0} \alpha_k(n),$$

where, for each $k \geq 0$, $\alpha_k(n)$ is nonnegative or nonpositive for large n, $\sum_{n\geq 0} \alpha_k(n)x^n$ converges for all $|x| < 1$, and

$$\sum_{n\geq 0} \alpha_k(n) = \pm\infty.$$

Then

$$\sum_{n\geq 0} a_n x^n \approx \sum_{k\geq 0} f_k(x) \qquad \textit{as } x \to 1^-,$$

where

$$f_k(x) = \sum_{n\geq 0} \alpha_k(n)x^n.$$

Proof. If we fix $K \geq 1$ and write

$$a_n = \sum_{k=0}^{K-1} \alpha_k(n) + \epsilon_K(n)$$

then by assumption we have

$$\epsilon_K(n) \sim \alpha_K(n)$$

as $n \to \infty$. By Theorem 1 we may deduce that

$$\sum_{n\geq 0} \epsilon_K(n)x^n \sim \sum_{n\geq 0} \alpha_K(n)x^n$$

as $x \to 1^-$, or, in other words,

$$\sum_{n\geq 0} \epsilon_K(n)x^n = f_K(x) + o(f_K(x)),$$

and thus

$$\begin{aligned}
\sum_{n\geq 0} a_n x^n &= \sum_{n\geq 0} \left(\sum_{k=0}^{K-1} \alpha_k(n) + \epsilon_K(n) \right) x^n \\
&= \sum_{k=0}^{K-1} f_k(x) + \sum_{n\geq 0} \epsilon_K(n)x^n \\
&= \sum_{k=0}^{K} f_k(x) + o(f_K(x))
\end{aligned}$$

as $x \to 1^-$. As K is arbitrary the result follows. □

Corollary 3. *Suppose*

$$\sum_{k=0}^{n} a_k \sim \sum_{k=0}^{n} b_k \qquad \text{as } n \to \infty,$$

where $\sum_{k \le n} b_k$ *is nonnegative or nonpositive for large* n,

$$\sum_{n \ge 0} \left(\sum_{k=0}^{n} b_k \right) x^n$$

converges for all $|x| < 1$, *and*

$$\sum_{n \ge 0} \sum_{k=0}^{n} b_k = \infty.$$

Then

$$\sum_{n \ge 0} a_n x^n \sim \sum_{n \ge 0} b_n x^n \qquad \text{as } x \to 1^-.$$

Proof. We have

$$\begin{aligned} \sum_{n \ge 0} a_n x^n &= (1-x) \sum_{n \ge 0} \left(\sum_{k=0}^{n} a_k \right) x^n \\ &\sim (1-x) \sum_{n \ge 0} \left(\sum_{k=0}^{n} b_k \right) x^n \\ &= \sum_{n \ge 0} b_n x^n \end{aligned}$$

as $x \to 1^-$ by Theorem 1. $\square$

Corollary 4. *Suppose*

$$\sum_{k=0}^{n} a_k \sim b_n \qquad \text{as } n \to \infty$$

and

$$b_n - b_{n-1} \sim c_n \qquad \text{as } n \to \infty,$$

where c_n *is nonnegative or nonpositive for large* n, $\sum_{n \ge 0} c_n x^n$ *converges for all* $|x| < 1$, *and*

$$\sum_{n \ge 0} c_n = \infty.$$

Then

$$\sum_{n \ge 0} a_n x^n \sim \sum_{n \ge 0} c_n x^n \qquad \text{as } x \to 1^-.$$

Proof. By the Stolz-Cesàro theorem we know that

$$\sum_{k=0}^{n} c_k \sim b_n \sim \sum_{k=0}^{n} a_k \qquad \text{as } n \to \infty, \tag{5}$$

so we may apply Corollary 3 to see that

$$\sum_{n \geq 0} a_n x^n \sim \sum_{n \geq 0} c_n x^n \qquad \text{as } x \to 1^-.$$

□

2 Examples

The examples below deal mainly with arithmetic functions from number theory since they're canonically so badly behaved. With the help of the results obtained above we can relate the behavior of their generating functions to power series with much more regular coefficients.

2.1 The divisor functions

Let

$$\sigma_a(n) = \sum_{d|n} d^a$$

represent the sum of a^{th} powers of the divisors of n. It is known that

$$\sum_{k=1}^{n} \sigma_0(k) \sim n \log n \qquad \text{as } n \to \infty$$

and

$$\sum_{k=1}^{n} \sigma_a(k) \sim \frac{\zeta(a+1)}{a+1} n^{a+1} \qquad \text{as } n \to \infty$$

for all $a \geq 1$. Since

$$n \log n - (n-1)\log(n-1) \sim \log n \qquad \text{as } n \to \infty$$

and

$$n^{a+1} - (n-1)^{a+1} \sim (a+1)n^a \qquad \text{as } n \to \infty,$$

we gain from Corollary 4 that

$$\sum_{n \geq 1} \sigma_0(n) x^n \sim \sum_{n \geq 2} (\log n) x^n \qquad \text{as } x \to 1^-$$

and

$$\sum_{n \geq 1} \sigma_a(n) x^n \sim \zeta(a+1) \sum_{n \geq 0} n^a x^n \qquad \text{as } x \to 1^-$$

for all $a \geq 1$.

2.2 The sum of two squares function

Let $r(n)$ denote the number of representations of n by two squares. It is known that

$$\sum_{k=1}^{n} r(k) \sim \pi n \qquad \text{as } n \to \infty,$$

so, since $\pi n - \pi(n-1) = \pi$, by Corollary 4 we have

$$\sum_{n \geq 1} r(n) x^n \sim \pi \sum_{n \geq 0} x^n = \frac{\pi}{1-x} \qquad \text{as } x \to 1^-.$$

2.3 Prime numbers

Let $\pi(n)$ denote the number of primes less than or equal to n. The prime number theorem says that

$$\pi(n) \sim \frac{n}{\log n},$$

so by Theorem 1 we have

$$\sum_{n \geq 1} \pi(n) x^n \sim \sum_{n \geq 1} \frac{n}{\log n} x^n \qquad \text{as } x \to 1^-.$$

We can also consider the power series having prime powers,

$$\sum_{p \text{ prime}} x^p = \sum_{n \geq 0} a_n x^n,$$

where

$$a_n = \begin{cases} 1 & \text{if } n \text{ is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

Of course

$$\sum_{k=0}^{n} a_k = \pi(n) \sim \frac{n}{\log n},$$

so since

$$\frac{n}{\log n} - \frac{n-1}{\log(n-1)} \sim \frac{1}{\log n}$$

we have, by Corollary 4,

$$\sum_{p \text{ prime}} x^p \sim \sum_{n \geq 1} \frac{x^n}{\log n} \qquad \text{as } x \to 1^-.$$

3 Explicit asymptotics via integrals

Lemma 5. *For a given function $\psi : [N, \infty) \to \mathbb{R}^+$ suppose that the map $t \mapsto \psi(t)x^t$ is unimodal with maximum at $t = t_x \geq N$, $0 < x < 1$. Then*

$$\sum_{n \geq N} \psi(n)x^n = \int_N^\infty \psi(t)x^t \, dt + O\left(\psi(t_x)x^{t_x}\right) + O(1)$$

as $x \to 1^-$.

Proof. For $N \leq t \leq t_x$ the map $t \mapsto \psi(t)x^t$ is monotone increasing, so

$$\begin{aligned}
\psi(N)x^N - \psi(t_x)x^{t_x} + \int_N^{t_x} \psi(t)x^t \, dt &\leq \psi(N)x^N - \int_{\lfloor t_x \rfloor}^{t_x} \psi(t)x^t \, dt + \int_N^{t_x} \psi(t)x^t \, dt \\
&= \psi(N)x^N + \int_N^{\lfloor t_x \rfloor} \psi(t)x^t \, dt \\
&= \psi(N)x^N + \sum_{n=N+1}^{\lfloor t_x \rfloor} \int_{n-1}^{n} \psi(t)x^t \, dt \\
&\leq \psi(N)x^N + \sum_{n=N+1}^{\lfloor t_x \rfloor} \int_{n-1}^{n} \psi(n)x^n \, dt \\
&= \sum_{N \leq n \leq t_x} \psi(n)x^n
\end{aligned}$$

and

$$\begin{aligned}
\psi(t_x)x^{t_x} + \int_N^{t_x} \psi(t)x^t \, dt &\geq \psi(t_x)x^{t_x} + \int_N^{\lfloor t_x \rfloor} \psi(t)x^t \, dt \\
&= \psi(t_x)x^{t_x} + \sum_{n=N+1}^{\lfloor t_x \rfloor} \int_{n-1}^{n} \psi(t)x^t \, dt \\
&\geq \psi(t_x)x^{t_x} + \sum_{n=N+1}^{\lfloor t_x \rfloor} \int_{n-1}^{n} \psi(n-1)x^{n-1} \, dt \\
&\geq \psi(\lfloor t_x \rfloor)x^{\lfloor t_x \rfloor} + \sum_{n=N+1}^{\lfloor t_x \rfloor} \psi(n-1)x^{n-1} \\
&= \sum_{N \leq n \leq t_x} \psi(n)x^n.
\end{aligned}$$

It follows that

$$\sum_{N \leq n \leq t_x} \psi(n)x^n = \int_N^{t_x} \psi(t)x^t \, dt + O(1) + O\big(\psi(t_x)x^{t_x}\big) \tag{6}$$

since $\psi(N)x^N = O(1)$ for $0 < x < 1$.

Similarly the map $t \mapsto \psi(t)x^t$ is monotone decreasing for $t \geq t_x$, so

$$\begin{aligned}
&-\psi(t_x)x^{t_x} + \int_{t_x}^{\infty} \psi(t)x^t\,dt \\
&\quad = -\psi(t_x)x^{t_x} + \int_{t_x}^{\lfloor t_x \rfloor + 1} \psi(t)x^t\,dt + \sum_{n \geq \lfloor t_x \rfloor + 1} \int_n^{n+1} \psi(t)x^t\,dt \\
&\quad \leq -\psi(t_x)x^{t_x} + \psi(t_x)x^{t_x} + \sum_{n \geq \lfloor t_x \rfloor + 1} \int_n^{n+1} \psi(n)x^n\,dt \\
&\quad = \sum_{n > t_x} \psi(n)x^n
\end{aligned}$$

and

$$\begin{aligned}
\psi(t_x)x^{t_x} + \int_{t_x}^{\infty} \psi(t)x^t\,dt &\geq \psi(t_x)x^{t_x} + \int_{\lfloor t_x \rfloor + 1}^{\infty} \psi(t)x^t\,dt \\
&= \psi(t_x)x^{t_x} + \sum_{n \geq \lfloor t_x \rfloor + 2} \int_{n-1}^{n} \psi(t)x^t\,dt \\
&\geq \psi(t_x)x^{t_x} + \sum_{n \geq \lfloor t_x \rfloor + 2} \int_{n-1}^{n} \psi(n)x^n\,dt \\
&\geq \psi(\lfloor t_x \rfloor + 1)x^{\lfloor t_x \rfloor + 1} + \sum_{n \geq \lfloor t_x \rfloor + 2} \psi(n)x^n \\
&= \sum_{n > t_x} \psi(n)x^n.
\end{aligned}$$

Thus

$$\sum_{n > t_x} \psi(n)x^n = \int_{t_x}^{\infty} \psi(t)x^t\,dt + O\big(\psi(t_x)x^{t_x}\big) \tag{7}$$

Combining (6) and (7) we have

$$\begin{aligned}
\sum_{n \geq N} \psi(n)x^n &= \sum_{N \leq n \leq t_x} \psi(n)x^n + \sum_{n > t_x} \psi(n)t^n \\
&= \int_N^{\infty} \psi(t)x^t\,dt + O(1) + O\big(\psi(t_x)x^{t_x}\big)\,,
\end{aligned}$$

as desired. □

3.1 The divisor functions again

We showed above that

$$\sum_{n \geq 1} \sigma_0(n)x^n \sim \sum_{n \geq 1} (\log n)x^n \qquad \text{as } x \to 1^-$$

and

$$\sum_{n\geq 1} \sigma_a(n)x^n \sim \zeta(a+1)\sum_{n\geq 0} n^a x^n \qquad \text{as } x \to 1^-$$

for all $a \geq 1$.

Let's consider the first sum. By differentiating we see that the map $t \mapsto (\log t)x^t$ has a maximum at

$$t_x = -\frac{1}{(\log x)W(-1/\log x)},$$

where W is the Lambert W function. Since $\log x \sim x-1$ as $x \to 1$ and $W(\lambda) \sim \log\lambda$ as $\lambda \to \infty$ we know that

$$(\log t_x)x^{t_x} \sim -\log(1-x)$$

as $x \to 1^-$. As for the corresponding integral, if we set $\lambda = -1/\log x$ then we have

$$\begin{aligned}\int_1^\infty (\log t)x^t\,dt &= \int_2^\infty e^{-t/\lambda}\log t\,dt \\ &= \lambda\int_{1/\lambda}^\infty e^{-s}\log(\lambda s)\,ds \\ &= \lambda\log\lambda\int_{1/\lambda}^\infty e^{-s}\,ds + \lambda\int_{1/\lambda}^\infty e^{-s}\log s\,ds,\end{aligned}$$

where in the second line we made the change of variables $t = \lambda s$. Both integrals converge as $\lambda \to \infty$, so the second term is clearly dominated by the first. Thus

$$\int_1^\infty (\log t)x^t\,dt \sim \lambda\log\lambda \sim \frac{\log(1-x)}{x-1}.$$

as $x \to 1^-$ so we may conclude by Lemma 5 that

$$\sum_{n\geq 1}\sigma_0(n)x^n \sim \frac{\log(1-x)}{x-1} \qquad \text{as } x \to 1^-. \tag{8}$$

Now let's consider the second sum. By differentiating we similarly find that the map $t \mapsto t^a x^t$ has a maximum at

$$t_x = -\frac{a}{\log x}$$

with height

$$t_x^a x^{t_x} = \frac{a^a e^{-a}}{(-\log x)^a}$$

The integral in this case is easier to handle since it can be evaluated in closed form as

$$\int_0^\infty t^a x^t\,dt = \frac{\Gamma(a+1)}{(-\log x)^{a+1}} \sim \frac{\Gamma(a+1)}{(1-x)^{a+1}}.$$

Appealing to Lemma 5 we may conclude that

$$\sum_{n\geq 1}\sigma_a(n)x^n \sim \frac{\zeta(a+1)\Gamma(a+1)}{(1-x)^{a+1}} \qquad \text{as } x \to 1^-. \tag{9}$$

3.2 Prime numbers again

We showed above that

$$\sum_{n\geq 1} \pi(n)x^n \sim \sum_{n\geq 1} \frac{n}{\log n}x^n \qquad \text{as } x \to 1^-$$

and

$$\sum_{p \text{ prime}} x^p \sim \sum_{n\geq 1} \frac{x^n}{\log n} \qquad \text{as } x \to 1^-.$$

The map $t \mapsto \frac{t}{\log t}x^t$ is unimodal on $t \geq 3$ for x sufficiently close to 1 with $x < 1$, and using an argument similar to the ones above it's possible to show that

$$\sum_{n\geq 1} \pi(n)x^n \sim -\frac{1}{(1-x)^2 \log(1-x)} \qquad \text{as } x \to 1^-.$$

The second sum is easier since the map $t \mapsto \frac{1}{\log t}x^t$ is strictly decreasing for all $t \geq 2$. We can then compare the sum to the integral immediately and, after some estimation, arrive at the conclusion that

$$\sum_{p \text{ prime}} x^p \sim \frac{1}{(x-1)\log(1-x)} \qquad \text{as } x \to 1^-.$$